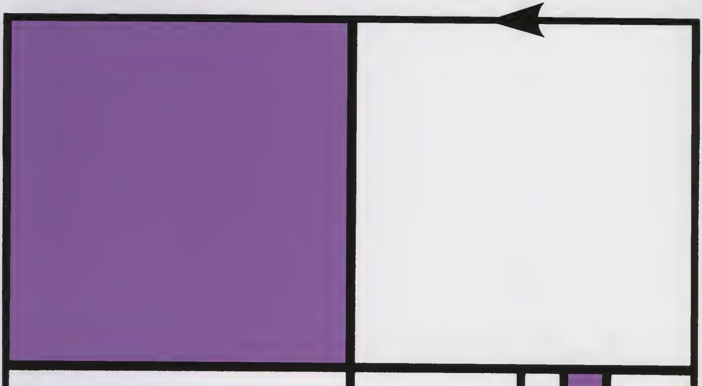
M337 Unit B3
Mathematics: A Third Level Course

The Open University

COMPLEX ANALYSIS

UNIT B3 TAYLOR SERIES



ANALYSIS COMPLEX ANALYSIS COMPLEX AN

M337 Unit B3
Mathematics: A Third Level Course

The Open University

COMPLEX ANALYSIS

UNIT B3 TAYLOR SERIES

Prepared by the Course Team

MPLEX ANALYSIS COMPLEX ANALYSIS COMPL

Before working through this text, make sure that you have read the *Course Guide* for M337 Complex Analysis.

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published 1993. Reprinted 1995, 1998, 2002, 2006.

Edited, designed and typeset by the Open University using the Open University TEX System.

Printed in Malta by Gutenberg Press Limited.

ISBN 0 7492 2181 X

This text forms part of an Open University Third Level Course. If you would like a copy of *Studying with The Open University*, please write to the Central Enquiry Service, PO Box 200, The Open University, Walton Hall, Milton Keynes, MK7 6YZ. If you have not already enrolled on the Course and would like to buy this or other Open University material, please write to Open University, Educational Enterprises Ltd, 12 Cofferidge Close, Stony Stratford, Milton Keynes, MK11 1BY, United Kingdom.

1.4

CONTENTS

Introduction 4

Study guide 4

1 Complex Series 5

1.1 Convergent series 5

1.2 Some basic series 8

1.3 Convergence theorems 10

1.4 Absolute convergence 15

2 Power Series 19

2.1 The radius of convergence 19

2.2 The disc of convergence 24

2.3 Differentiation of power series 26

3 Taylor's Theorem 30

3.1 Taylor series 30

3.2 Basic Taylor series 34

3.3 Proof of Taylor's Theorem and Cauchy's nth Derivative Formula 36

4 Manipulating Taylor Series 39

4.1 Finding Taylor series (audio-tape) 39

4.2 The radius of convergence of a Taylor series 44

5 The Uniqueness Theorem 45

5.1 Zeros of a function 46

5.2 Using power series to define functions 51

Exercises 53

Solutions to the Problems 55

Solutions to the Exercises 61

INTRODUCTION

In this unit we develop an important technique for investigating the properties of analytic functions. Rather than investigate a given analytic function directly, we represent the function by an expression known as a *power series* and carry out the investigation on the series instead.

A power series can be thought of as an infinite degree polynomial, such as

$$1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots .$$

In order to make this notion of a power series precise, we need to be able to work with sums that involve infinitely many terms. In Section 1 we show how this is done by using the idea of the limit of an infinite sequence to give a rigorous definition of an infinite sum, or a *convergent series* as it is more formally known. We then discuss a number of tests that can be used to check whether a given series is convergent.

In Section 2 we give a formal definition of a power series, and observe that any power series defines a function. For example, we can define a function f by the rule

$$f(z) = 1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots .$$

We then show that we can find the derivative of such a function by differentiating the power series term by term as if it were a polynomial.

In Section 3 we turn the process around. Instead of starting with a complex power series and using it to define a function, we start with a function f which is analytic at a point α and obtain a power series that represents f near α. This is the *Taylor series* for f about α, a version of which may be familiar to you from real analysis. The existence and uniqueness of the Taylor series æfor f about α is established in Taylor's Theorem. After proving this theorem, we give the proof of Cauchy's nth Derivative Formula, promised in *Unit B2*.

In Section 4 we give a number of techniques which are useful for calculating Taylor series for particular functions.

Finally, in Section 5, we present a uniqueness theorem which shows that remarkably little information is required to identify an analytic function. Just as Cauchy's Integral Formula tells us that the values of an analytic function within a circle are uniquely determined by the values of the function on the circle, so the Uniqueness Theorem tells us that the values of an analytic function on a region $\mathcal{R}$ are uniquely determined by the values of the function on a sequence of points converging to a point of $\mathcal{R}$.

Study guide

This unit is an important one, as it forms the basis for *Unit B4* on Laurent series. The most important sections are Sections 2 and 3 — especially Section 3.

Section 1 is slightly longer than the other sections, so if you have not already studied series in a course on real analysis you should be prepared to spend extra time on this part of the unit.

If you have studied real series before, then you will be familiar with the form of many of the results in this unit. There are, however, important differences between the real and complex results and you should read the unit with this in mind. For example, in Section 2 the interval of convergence is replaced by the disc of convergence and, in Section 3, the fact that an analytic function f has derivatives of all orders implies that the Taylor series for f is automatically an infinite series.

Section 4 is the audio-tape section, and it illustrates how you can find new Taylor series by manipulating other known Taylor series. This approach is often easier than a direct application of Taylor's Theorem.

Section 5 contains some theoretical work on uniqueness that prepares the way for the work on analytic continuation in *Unit C3*. The final subsection indicates an alternative approach to the definition of complex functions which you may find interesting if you have the time.

Finally, note that this unit contains several applications of the Monotone Convergence Theorem for real sequences, which was stated in *Unit A3*, Subsection 5.2. For convenience, we restate it here.

Monotone Convergence Theorem If the real sequence $\{a_n\}$ is increasing and bounded above, or decreasing and bounded below, then $\{a_n\}$ is convergent.

1 COMPLEX SERIES

After working through this section, you should be able to:

(a) explain what is meant by *convergent* and *divergent* series;
(b) use partial sums to find the *sum* of a given convergent series;
(c) use the Non-null Test to show that a given series is divergent;
(d) recognize a convergent geometric series, and write down its sum;
(e) use the Combination Rules to find the sum of a convergent series;
(f) check the convergence of a series by inspecting its real and imaginary parts;
(g) check for convergence using the Comparison Test;
(h) explain what is meant by an *absolutely convergent* series;
(i) check for absolute convergence using the Absolute Convergence and Ratio Tests.

1.1 Convergent series

Given any finite collection of complex numbers, we can form their sum. In this section we investigate whether this idea of summation can be generalized to an infinite sequence of complex numbers $\{z_n\}$. Roughly speaking, the idea is to see whether successive sums $z_1, z_1 + z_2, z_1 + z_2 + z_3, \ldots$ (called *partial sums*) approach a limit as more and more terms are included. In preparation for this, we make the following definitions.

Definitions Given a sequence $\{z_n\}$ of complex numbers, the expression

$$z_1 + z_2 + z_3 + \cdots$$

is called an **infinite series**, or simply a **series**. The number z_n is called the **nth term** of the series.

The **nth partial sum** of the series is

$$s_n = z_1 + z_2 + \cdots + z_n = \sum_{k=1}^{n} z_k.$$

If each $z_n \in \mathbb{R}$, then the series is a *real series*.

Remarks

1 Note that in the definition of the nth partial sum we have used the variable k. The letter n is not available and, in complex analysis, we prefer to avoid i.

2 Note carefully the difference between the sequence of *terms* $\{z_n\}$ of the series and the sequence of *partial sums* $\{s_n\}$ of the series.

As for finite sums we frequently use sigma notation to represent an infinite series. Thus we write

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots .$$

If we need to begin a series with a term other than z_1, then we write, for example,

$$\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + \cdots \quad \text{or} \quad \sum_{n=3}^{\infty} z_n = z_3 + z_4 + z_5 + \cdots .$$

For such a series, the nth partial sum s_n is defined to be the sum of all the terms up to and including z_n. For instance, the 5th partial sum of the series $\sum_{n=3}^{\infty} z_n$ is

$$s_5 = \sum_{k=3}^{5} z_k = z_3 + z_4 + z_5.$$

Here the first and second partial sums are not defined.

Problem 1.1

Evaluate the 0th, 1st, 2nd and 3rd partial sums of the series $\sum_{n=0}^{\infty} \left(\frac{i}{2}\right)^n$.

It is occasionally possible to find an explicit expression for the nth partial sum of a series. For example, the nth partial sum of the series in Problem 1.1 is given by the formula for the sum of a finite geometric series:

$$s_n = \frac{1 - \left(\frac{i}{2}\right)^{n+1}}{1 - \frac{i}{2}}.$$

The sum of the finite geometric series $1 + z + z^2 + \cdots + z^n$ is equal to

$$\frac{1 - z^{n+1}}{1 - z} \quad (z \neq 1).$$

(See the remark that follows Theorem 1.3 in *Unit A1*.)

Since the sequence $\{(i/2)^n\}$ is null, this sequence of partial sums tends to $1/(1 - i/2)$ as $n \to \infty$. Thus it seems reasonable to define the sum of the series in Problem 1.1 to be $1/(1 - i/2)$. More generally, we make the following definition.

> **Definitions** The (complex) series
>
> $$z_1 + z_2 + z_3 + \cdots$$
>
> is **convergent** with **sum** s if the sequence $\{s_n\}$ of partial sums converges to s. In this case, we say that the series **converges** to s, and write
>
> $$z_1 + z_2 + z_3 + \cdots = s, \quad \text{or} \quad \sum_{n=1}^{\infty} z_n = s.$$
>
> The series **diverges** if the sequence $\{s_n\}$ diverges.

Similarly, a series that begins with a term other than z_1 has a sum s if its partial sums converge to s. We then write, for example,

$$z_0 + z_1 + z_2 + \cdots = s,$$

or

$$\sum_{n=0}^{\infty} z_n = s.$$

Thus we can prove results about the *convergence of a series* $\sum_{n=1}^{\infty} z_n$ by applying *convergence results for sequences* to the sequence $\{s_n\}$ of partial sums.

Example 1.1

For each of the following infinite series, calculate the nth partial sum, and determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{\infty} 2i = 2i + 2i + 2i + \cdots$

(b) $\sum_{n=0}^{\infty} 2\left(\frac{i}{3}\right)^n = 2 + 2\left(\frac{i}{3}\right) + 2\left(\frac{i}{3}\right)^2 + 2\left(\frac{i}{3}\right)^3 + \cdots$

Solution

(a) In this case,

$$s_n = \sum_{k=1}^{n} 2i = 2ni.$$

Since the sequence $\{2ni\}$ does not converge (indeed it tends to ∞), the series $\sum_{n=1}^{\infty} 2i$ is divergent.

(b) By the formula for the sum of a finite geometric series, we obtain

$$\begin{aligned} s_n &= 2\left(1 + \frac{i}{3} + \left(\frac{i}{3}\right)^2 + \cdots + \left(\frac{i}{3}\right)^n\right) \\ &= 2\frac{1 - \left(\frac{i}{3}\right)^{n+1}}{1 - \frac{i}{3}} \\ &= \frac{6}{3-i}\left(1 - \left(\frac{i}{3}\right)^{n+1}\right). \end{aligned}$$

Since $\{(i/3)^n\}$ is a null sequence, it follows that

$$\lim_{n\to\infty} s_n = \frac{6}{3-i} = \frac{6(3+i)}{10} = \frac{1}{5}(9+3i),$$

and so the series converges, with sum $\frac{1}{5}(9+3i)$. ■

Problem 1.2

For each of the following series, calculate the nth partial sum s_n, and determine whether the series is convergent or divergent. If the series converges calculate its sum.

(a) $\sum_{n=1}^{\infty} \left(\frac{i}{4}\right)^n$ (b) $\sum_{n=1}^{\infty} 7(-i)^n$ (c) $\sum_{n=1}^{\infty} \left(\frac{1-i}{2}\right)^n$

It is sometimes possible to show that a series converges or diverges without having to calculate its partial sums. One technique for proving divergence relies on the following theorem.

Theorem 1.1 If $\sum_{n=1}^{\infty} z_n$ is a convergent series, then $\{z_n\}$ is a null sequence.

Proof Let $s_n = \sum_{k=1}^{n} z_k$ be the nth partial sum of $\sum_{n=1}^{\infty} z_n$. Then $\{s_n\}$ is a convergent sequence, with limit s (say). Since

$$s_n = s_{n-1} + z_n, \qquad \text{for } n = 2, 3, \ldots,$$

we have

$$z_n = s_n - s_{n-1}, \qquad \text{for } n = 2, 3, \ldots,$$

and so, by the Combination Rules for sequences (*Unit A3*, Theorem 1.3),

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

Thus $\{z_n\}$ is a null sequence. ■

Theorem 1.1 shows that a series cannot converge unless its terms form a null sequence. We therefore have the following divergence test.

Corollary Non-null Test

If the sequence $\{z_n\}$ is not null, then the series $\sum_{n=1}^{\infty} z_n$ is divergent.

A similar result holds for series beginning with a term other than z_1.

For example, the series

$$\sum_{n=1}^{\infty} ni = i + 2i + 3i + \cdots$$

is divergent, since the sequence $\{ni\}$ is not null (it tends to infinity).

It is important to realize that the *converse* of the Non-null Test is false. For example, the sequence $\{1/n\}$ is null and yet the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots,$$

is divergent. This series is called the **harmonic series** and in Theorem 1.3 you will see that it diverges.

Problem 1.3

Use the Non-null Test to show that each of the following series is divergent.

(a) $\sum_{n=1}^{\infty} (1+i)^n$ (b) $\sum_{n=1}^{\infty} i(-1)^n$ (c) $\sum_{n=1}^{\infty} n!$ (d) $\sum_{n=1}^{\infty} \frac{n^2+i}{2n^2+n+3}$

1.2 Some basic series

Many of the series we have mentioned so far have been of the form

$$\sum_{n=0}^{\infty} az^n = a + az + az^2 + \cdots .$$

Such series are known as **geometric series** with **common ratio** z. The following theorem enables us to decide whether a given geometric series is convergent or divergent.

Theorem 1.2 Geometric Series

(a) If $|z| < 1$ and $a \in \mathbb{C}$, then the series $\sum_{n=0}^{\infty} az^n$ is convergent with sum $a/(1-z)$.

(b) If $|z| \geq 1$ and $a \in \mathbb{C} - \{0\}$, then the series $\sum_{n=0}^{\infty} az^n$ is divergent.

Note that different values of z give rise to different geometric series and that the sum in part (a) depends on the value of z. However, it is convenient to talk about *the* series $\sum_{n=0}^{\infty} az^n$.

Proof

(a) If $z \neq 1$, then the nth partial sum is given by

$$s_n = a + az + az^2 + \cdots + az^n = \frac{a(1-z^{n+1})}{1-z}.$$

Furthermore, if $|z| < 1$, then $\{z^n\}$ is a basic null sequence. So, by the Combination Rules for sequences,

Unit A3, Theorem 1.2(b)

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-z^{n+1})}{1-z} = \frac{a}{1-z}\left(1 - z\left(\lim_{n\to\infty} z^n\right)\right) = \frac{a}{1-z}.$$

Thus, the series $\sum_{n=0}^{\infty} az^n$ is convergent with sum $a/(1-z)$.

(b) If $|z| \geq 1$, then $\{z^n\}$ does not converge to 0. Since $a \neq 0$, it follows that the sequence $\{az^n\}$ does not converge to 0, either. The result follows from the Non-null Test. ■

$\{z^n\}$ tends to ∞ if $|z| > 1$; it diverges if $|z| = 1$, $z \neq 1$; and it converges to 1 if $z = 1$ (*Unit A3*, Theorem 1.7).

Problem 1.4

Use Theorem 1.2 to check your answers to Problem 1.2.

The convergence or divergence of a given complex series is often investigated by comparing it with another series that is known to converge or diverge. As you will see later in this section, such comparisons are frequently made with geometric series. Other series that are sometimes used in this way are those of the form

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots,$$

where p is real. The convergence of such series depends on the value of p.

Theorem 1.3 The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots,$$

converges if $p > 1$, and diverges if $p \leq 1$.

For example, if $p = 2$, then we obtain the convergent series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots,$$

You will see later in the course that this series converges to $\pi^2/6$.

whereas, if $p = 1$, we obtain the divergent series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

This is the harmonic series that we mentioned earlier.

Proof If $p \leq 0$ then all the terms of the series are greater than or equal to 1, so the series diverges, by the Non-null Test. This leaves two other cases to consider: $0 < p \leq 1$, and $p > 1$.

For the case where $0 < p \leq 1$, the nth partial sum of the series is equal to the total area of the rectangles shown in Figure 1.1. (For clarity the vertical axis has been stretched.) This area is greater than the area under the graph of the real function $f(x) = 1/x^p$ between 1 and $n+1$, so

$$s_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} > \int_1^{n+1} \frac{1}{x^p}\,dx.$$

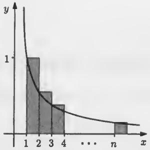


Figure 1.1
$f(x) = 1/x^p, 0 < p \leq 1$

Unit B1, Frame 6

Since $1/x^p \geq 1/x$, we can use the Monotonicity Inequality for integrals to obtain

$$s_n > \int_1^{n+1} \frac{1}{x^p}\,dx \geq \int_1^{n+1} \frac{1}{x}\,dx = \log_e(n+1).$$

It follows that the sequence of partial sums $\{s_n\}$ tends to infinity, and so the series diverges for $p \leq 1$.

For the case where $p > 1$, the nth partial sum is equal to the total area of the rectangles *beneath* the graph of the real function $f(x) = 1/x^p$ in Figure 1.2. If we remove the first rectangle by subtracting 1, then it follows that

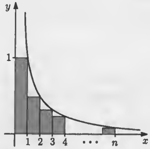


Figure 1.2
$f(x) = 1/x^p, p > 1$

$$s_n - 1 = \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} < \int_1^{n} \frac{1}{x^p}\,dx$$
$$= \left[\frac{x^{1-p}}{1-p}\right]_1^n = \left(\frac{n^{1-p}}{1-p} - \frac{1}{1-p}\right) < \frac{1}{p-1}.$$

So the sequence of partial sums $\{s_n\}$ is increasing and bounded above by $1 + 1/(p-1)$. By the Monotone Convergence Theorem (see the Introduction), the partial sums, and hence the series, are convergent. ■

1.3 Convergence theorems

When we investigate the convergence properties of series, it is not always possible, or convenient, to work directly with the partial sums. You have already seen that the Non-null Test can sometimes be used to check *divergence* without the need to calculate partial sums, so let us now turn our attention to some rules for checking *convergence*.

We begin with the Combination Rules.

Theorem 1.4 Combination Rules

If $\sum_{n=1}^{\infty} z_n = s$ and $\sum_{n=1}^{\infty} w_n = t$, then

Sum Rule $\quad \sum_{n=1}^{\infty} (z_n + w_n) = s + t;$

Multiple Rule $\quad \sum_{n=1}^{\infty} \lambda z_n = \lambda s,\;$ for $\lambda \in \mathbb{C}$.

Again, a similar result holds for series beginning with terms other than z_1 and w_1.

For example, we know from Problem 1.2 that

$$\sum_{n=1}^{\infty} \left(\frac{i}{4}\right)^n = \frac{-1+4i}{17} \quad \text{and} \quad \sum_{n=1}^{\infty} \left(\frac{1-i}{2}\right)^n = -i.$$

Here $s = \sum_{n=1}^{\infty} \left(\frac{i}{4}\right)^n$ and $t = \sum_{n=1}^{\infty} \left(\frac{1-i}{2}\right)^n$.

It follows that

$$\sum_{n=1}^{\infty} \left(17\left(\frac{i}{4}\right)^n + i\left(\frac{1-i}{2}\right)^n\right) = (-1+4i) + i(-i) = 4i.$$

Proof The proof of the Combination Rules for series follows from the corresponding Combination Rules for sequences. We shall prove the Sum Rule, and leave you to prove the Multiple Rule in Problem 1.5.

We have

$$\begin{aligned}\sum_{k=1}^{\infty}(z_k + w_k) &= \lim_{n\to\infty}\sum_{k=1}^{n}(z_k + w_k)\\ &= \lim_{n\to\infty}\left(\sum_{k=1}^{n} z_k + \sum_{k=1}^{n} w_k\right)\\ &= \lim_{n\to\infty}\sum_{k=1}^{n} z_k + \lim_{n\to\infty}\sum_{k=1}^{n} w_k \qquad \text{(Sum Rule for sequences)}\\ &= s + t. \quad \blacksquare\end{aligned}$$

Problem 1.5

Prove the Multiple Rule for series.

An immediate consequence of the Combination Rules is that if $\{z_n\}$ is a sequence for which both the series $\sum_{n=1}^{\infty} \operatorname{Re} z_n$ and $\sum_{n=1}^{\infty} \operatorname{Im} z_n$ converge, then the series $\sum_{n=1}^{\infty} z_n$ converges and

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \operatorname{Re} z_n + i\sum_{n=1}^{\infty} \operatorname{Im} z_n.$$

In fact the converse is also true. For if the series $\sum_{n=1}^{\infty} z_n$ converges to s, say, then

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n} \operatorname{Re} z_k &= \lim_{n\to\infty} \operatorname{Re}\left(\sum_{k=1}^{n} z_k\right)\\ &= \operatorname{Re} s.\end{aligned}$$

Here we have used the result that the real part of the limit of a sequence is the limit of the real parts: see *Unit A3*, Theorem 1.4(c).

So $\sum_{n=1}^{\infty} \operatorname{Re} z_n$ converges to $\operatorname{Re} s$.

A similar argument shows that $\sum_{n=1}^{\infty} \operatorname{Im} z_n$ converges to $\operatorname{Im} s$.

The series $\sum_{n=1}^{\infty} \operatorname{Re} z_n$ and $\sum_{n=1}^{\infty} \operatorname{Im} z_n$ are called, respectively, the **real part** and **imaginary part** of the series $\sum_{n=1}^{\infty} z_n$.

We have therefore established the following equivalence between the convergence of a series and the convergence of its real and imaginary parts.

> **Theorem 1.5** The series $\sum_{n=1}^{\infty} z_n$ is convergent if and only if both the series $\sum_{n=1}^{\infty} \operatorname{Re} z_n$ and $\sum_{n=1}^{\infty} \operatorname{Im} z_n$ are convergent. In this case
>
> $$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \operatorname{Re} z_n + i\sum_{n=1}^{\infty} \operatorname{Im} z_n.$$

This theorem can be used to check whether a series converges *or* diverges. For example, the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+i\frac{1}{n}\right)=(1+i)+(\tfrac{1}{4}+\tfrac{1}{2}i)+(\tfrac{1}{9}+\tfrac{1}{3}i)+\cdots$$

diverges because its imaginary part $\sum_{n=1}^{\infty} 1/n$ is divergent. On the other hand, the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+i\frac{1}{n^3}\right)=(1+i)+(\tfrac{1}{4}+\tfrac{1}{8}i)+(\tfrac{1}{9}+\tfrac{1}{27}i)+\cdots$$

converges because both $\sum_{n=1}^{\infty} 1/n^2$ and $\sum_{n=1}^{\infty} 1/n^3$ converge.

Theorem 1.5 can also be used in a rather unexpected way to find the sums of certain *real* series.

Example 1.2

Let x be an arbitrary real number. Find the sum of the series

$$\sum_{n=0}^{\infty}\frac{1}{2^n}\sin nx.$$

Note that different values of x give rise to different series.

Solution

First observe that

$$\sin nx = \mathrm{Im}(e^{inx}), \qquad \text{for } n=0,1,2,\ldots.$$

Hence

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{2^n}\sin nx &= \mathrm{Im}\left(\sum_{n=0}^{\infty}\frac{1}{2^n}e^{inx}\right) && \text{(Theorem 1.5)}\\ &= \mathrm{Im}\left(\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{ix}\right)^n\right)\\ &= \mathrm{Im}\left(\frac{1}{1-\frac{1}{2}e^{ix}}\right) && \text{(Theorem 1.2(a))}\\ &= \mathrm{Im}\left(\frac{1-\frac{1}{2}e^{-ix}}{(1-\frac{1}{2}e^{ix})(1-\frac{1}{2}e^{-ix})}\right)\\ &= \mathrm{Im}\left(\frac{(1-\frac{1}{2}\cos x)+\frac{1}{2}i\sin x}{1+\frac{1}{4}-\frac{1}{2}(e^{ix}+e^{-ix})}\right)\\ &= \frac{2\sin x}{5-4\cos x}. \quad \blacksquare\end{aligned}$$

Here we are multiplying the numerator and denominator by $1-\frac{1}{2}e^{-ix}$, the conjugate of $1-\frac{1}{2}e^{ix}$.

Problem 1.6

Find the sum of the series $\sum_{n=0}^{\infty}\frac{1}{2^n}\cos nx$.

We often need to know that a given series converges without needing to know the value of the sum. In such a case it may be possible to use a theorem like the following, which enables us to test a series for convergence by comparing it with another series that is known to converge.

Theorem 1.6 Comparison Test

If $\sum_{n=1}^{\infty} a_n$ is a real convergent series of positive terms, and if

$$|z_n| \le a_n, \qquad \text{for } n = 1, 2, \ldots,$$

then the series $\sum_{n=1}^{\infty} z_n$ is convergent.

$\{z_n\}$ is dominated by $\{a_n\}$.

Before giving a proof of this theorem we illustrate how it is used by reconsidering the series in Example 1.2. This method is simpler but it does not yield a value for the sum.

Example 1.3

Show that the following series is convergent.

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \sin nx, \qquad \text{where } x \in \mathbb{R}$$

Solution

First observe that

$$\left| \frac{1}{2^n} \sin nx \right| \le \frac{1}{2^n}, \qquad \text{for } n = 0, 1, 2, \ldots .$$

Now the series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is convergent, by Theorem 1.2(a), so the series

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \sin nx$$

is convergent, by the Comparison Test.

(Although the series includes a term corresponding to $n = 0$, this does not affect the convergence of the series. It is therefore legitimate to use the Comparison Test here.) ■

Problem 1.7

Show that the following series is convergent.

$$\sum_{n=1}^{\infty} \frac{\cos n}{n\sqrt{n}}$$

The next problem asks you to use the Comparison Test to prove a remarkable result about complex series that will be needed in the next subsection.

Problem 1.8

Use the Comparison Test to prove that the series $\sum_{n=1}^{\infty} z_n$ converges if the series $\sum_{n=1}^{\infty} |z_n|$ converges.

We end this subsection by giving the proof of Theorem 1.6.

Proof of the Comparison Test

Let $\sum_{n=1}^{\infty} z_n$ be a complex series, and let $\sum_{n=1}^{\infty} a_n$ be a real convergent series of positive terms satisfying $|z_n| \le a_n$, for $n = 1, 2 \dots$.

This proof may be omitted on a first reading.

We shall split the proof that the series $\sum_{n=1}^{\infty} z_n$ converges into three stages. First we prove it for the case where all the terms z_n are real and non-negative, then for the case where they are arbitrary real numbers, and finally for the case where they are arbitrary complex numbers.

(a) If all the z_ns are real and non-negative, then $0 \le z_n \le a_n$ for $n = 1, 2, \dots$.

So the partial sums $s_n = \sum_{k=1}^{n} z_k$ satisfy

$$s_n \le s_{n+1} \quad \text{and} \quad s_n \le \sum_{k=1}^{\infty} a_k, \qquad \text{for all } n = 1, 2, \dots.$$

Hence $\{s_n\}$ is an increasing real sequence, bounded above by the sum $\sum_{n=1}^{\infty} a_n$. So, by the Monotone Convergence Theorem, $\{s_n\}$ is convergent.

It follows that the series $\sum_{n=1}^{\infty} z_n$ is convergent.

(b) If all the z_ns are arbitrary real numbers, then we can separate the positive and negative terms by writing $z_n = z_n^+ - z_n^-$, where

$$z_n^+ = \begin{cases} z_n, & \text{if } z_n > 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad z_n^- = \begin{cases} -z_n, & \text{if } z_n < 0, \\ 0, & \text{otherwise.} \end{cases}$$

For example, if

$$\sum_{n=1}^{\infty} z_n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots,$$

then

$$\sum_{n=1}^{\infty} z_n^+ = 1 + 0 + \frac{1}{4} + 0 + \frac{1}{16} + \cdots$$

and

$$\sum_{n=1}^{\infty} z_n^- = 0 + \frac{1}{2} + 0 + \frac{1}{8} + 0 + \cdots.$$

Then

$$|z_n^+| \le |z_n| \le a_n \quad \text{and} \quad |z_n^-| \le |z_n| \le a_n,$$

and so, by part (a) of the proof,

$$\sum_{n=1}^{\infty} z_n^+ \quad \text{and} \quad \sum_{n=1}^{\infty} z_n^-$$

are convergent. Thus, by the Combination Rules, the series

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} (z_n^+ - z_n^-)$$

is also convergent.

(c) Finally, for arbitrary complex z_ns we have

$$|\operatorname{Re} z_n| \le |z_n| \le a_n \quad \text{and} \quad |\operatorname{Im} z_n| \le |z_n| \le a_n,$$

and so, by part (b) of the proof, the series

$$\sum_{n=1}^{\infty} \operatorname{Re} z_n \quad \text{and} \quad \sum_{n=1}^{\infty} \operatorname{Im} z_n$$

are convergent. It follows from Theorem 1.5 that the series $\sum_{n=1}^{\infty} z_n$ is convergent. ■

1.4 Absolute convergence

In Problem 1.8 you saw that the convergence of the series $\sum_{n=1}^{\infty} |z_n|$ is sufficient to ensure that the series $\sum_{n=1}^{\infty} z_n$ converges. This observation is useful, for it is often easier to check the convergence of a series like $\sum_{n=1}^{\infty} |z_n|$ of real positive terms, than to check the convergence of $\sum_{n=1}^{\infty} z_n$ directly. Of course, this can be done only for series where $\sum_{n=1}^{\infty} |z_n|$ converges. Such series are said to be *absolutely convergent*, or to *converge absolutely.*

Definition The (complex) series $\sum_{n=1}^{\infty} z_n$ is **absolutely convergent** if the real series $\sum_{n=1}^{\infty} |z_n|$ is convergent.

For example, if $|z| < 1$, then the geometric series $\sum_{n=1}^{\infty} az^n$ is absolutely convergent. Indeed, $\sum_{n=1}^{\infty} |az^n| = \sum_{n=1}^{\infty} |a||z|^n$ is convergent because it is a (real) geometric series with common ratio $|z|$ less than 1.

We can now write out the result of Problem 1.8 in the form of a convergence test.

Theorem 1.7 Absolute Convergence Test

If the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then the series $\sum_{n=1}^{\infty} z_n$ is convergent.

Example 1.4

Show that the following series is convergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} i^n}{n^3} = i - \frac{i^2}{2^3} + \frac{i^3}{3^3} - \frac{i^4}{4^3} + \frac{i^5}{5^3} - \cdots$$

Solution

The series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1} i^n}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

is convergent by Theorem 1.3. So, by the Absolute Convergence Test, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} i^n}{n^3}$$

is convergent. ■

The next example uses the Absolute Convergence Test to generalize the result in Theorem 1.3.

Example 1.5

Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots$ converges whenever $\operatorname{Re} z > 1$.

The series $\sum_{n=1}^{\infty} \frac{1}{n^z}$, for $\operatorname{Re} z > 1$, is used to define the Riemann zeta function, which is important in Number Theory (see *Unit C3*).

Solution

We prove that the series is absolutely convergent for $\operatorname{Re} z > 1$. The result then follows immediately from the Absolute Convergence Test. We have

$$\left|\frac{1}{n^z}\right| = \left|e^{-z \operatorname{Log} n}\right| = e^{\operatorname{Re}(-z \operatorname{Log} n)} = e^{-(\operatorname{Re} z)(\operatorname{Log} n)} = \frac{1}{n^{\operatorname{Re} z}}.$$

Recall that

$$|e^{\alpha}| = e^{\operatorname{Re} \alpha}$$

(*Unit A2*, Theorem 4.1(b)).

But if $\operatorname{Re} z > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^{\operatorname{Re} z}}$ converges, by Theorem 1.3, and so $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges absolutely. By the Absolute Convergence Test, $\sum_{n=1}^{\infty} \frac{1}{n^z}$ is convergent. ■

Problem 1.9

Determine which of the following series are absolutely convergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n(1+i)^n}{2^n} = 1 - \frac{(1+i)}{2} + \frac{(1+i)^2}{4} - \frac{(1+i)^3}{8} + \frac{(1+i)^4}{16} - \cdots$

The Absolute Convergence Test states that if the series $\sum_{n=1}^{\infty} |z_n|$ is convergent, then so is $\sum_{n=1}^{\infty} z_n$, but it does not indicate any connection between the sums of these two series. The following result fills this gap.

> **Theorem 1.8 Triangle Inequality**
>
> If the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then
>
> $$\left|\sum_{n=1}^{\infty} z_n\right| \le \sum_{n=1}^{\infty} |z_n|.$$

This is an infinite form of the Triangle Inequality of *Unit A1*, Section 5.

For example, the series $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$ is absolutely convergent because

$$\sum_{n=1}^{\infty} \left|\frac{i^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges by Theorem 1.3. So

$$\left|\sum_{n=1}^{\infty} \frac{i^n}{n^2}\right| \le \sum_{n=1}^{\infty} \frac{1}{n^2} \quad (= \pi^2/6, \text{as noted on page 9}).$$

Proof By the Triangle Inequality,

$$\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|, \qquad \text{for } n = 1, 2, \ldots,$$

so

$$\lim_{n\to\infty}\left|\sum_{k=1}^{n} z_k\right| \leq \lim_{n\to\infty}\sum_{k=1}^{n} |z_k|.$$

Here we are using a result from real analysis that if $a_n \leq b_n$, for $n = 1, 2, \ldots$, and

$$s = \lim_{n\to\infty} a_n, t = \lim_{n\to\infty} b_n$$

exist, then

$$s \leq t.$$

This can easily be proved from the definition of limit.

Hence

$$\begin{aligned}\left|\sum_{k=1}^{\infty} z_k\right| &= \left|\lim_{n\to\infty}\sum_{k=1}^{n} z_k\right| \\ &= \lim_{n\to\infty}\left|\sum_{k=1}^{n} z_k\right| \\ &\leq \lim_{n\to\infty}\sum_{k=1}^{n} |z_k| \\ &= \sum_{k=1}^{\infty} |z_k|. \quad \blacksquare\end{aligned}$$

Here we are using the fact that the modulus of a limit is the limit of the modulus: see *Unit A3*, Theorem 1.4(a).

It is important to realize that not every convergent series is absolutely convergent. For example, in Problem 1.9 you showed that the following series does *not* converge absolutely:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots,$$

and yet this series does converge, as we now ask you to show.

Problem 1.10

Use the identity

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$$

to prove that the following series converges.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}\cdots$$

Fortunately, many of the series we consider in this course *are* absolutely convergent, so we end this section with a test that can sometimes be used to check absolute convergence. It is based on the behaviour of the ratio between consecutive terms of a series.

Theorem 1.9 Ratio Test

Suppose that $\sum_{n=1}^{\infty} z_n$ is a complex series for which

$$\left|\frac{z_{n+1}}{z_n}\right| \to l \text{ as } n \to \infty.$$

(a) If $0 \leq l < 1$, then $\sum_{n=1}^{\infty} z_n$ is absolutely convergent.

(b) If $l > 1$, then $\sum_{n=1}^{\infty} z_n$ is divergent.

Remarks

1 The Ratio Test yields no information if $l = 1$.

2 The case $l > 1$ includes the situation where $\left|\frac{z_{n+1}}{z_n}\right| \to \infty$ as $n \to \infty$.

3 In applications of the Ratio Test, it may happen that some of the terms z_n take the value 0, in which case the ratio z_{n+1}/z_n is not defined. However, if this happens for at most finitely many n, then the limit of the sequence $\{|z_{n+1}/z_n|\}$ may still exist, and it may still be possible to apply the test.

Before giving a proof of the Ratio Test, we illustrate how it is used.

Example 1.6

Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n}$ converges absolutely if $|z| < 1$, but diverges if $|z| > 1$.

Solution

Let

$$z_n = \frac{(-1)^n z^n}{n}.$$

Be careful to distinguish between the term z_n and the power z^n.

The series is clearly absolutely convergent if $z = 0$, so let $z \neq 0$. Then we have

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(-1)^{n+1}z^{n+1}}{n+1}\right| \div \left|\frac{(-1)^n z^n}{n}\right| = \frac{n|z|}{n+1},$$

which tends to $|z|$ as $n \to \infty$. It follows from the Ratio Test, with $l = |z|$, that $\sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n}$ is absolutely convergent if $|z| < 1$, and divergent if $|z| > 1$. ■

Problem 1.11

(a) Use the Ratio Test to decide whether or not the series $\sum_{n=1}^{\infty} \frac{n^2}{3^n + i}$ is absolutely convergent.

(b) Prove that the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is absolutely convergent for all $z \in \mathbb{C}$.

Proof of the Ratio Test

This proof may be omitted on a first reading.

(a) For the case $0 \leq l < 1$, we choose $\varepsilon > 0$ so that

$$l + \varepsilon < 1.$$

For example, take $\varepsilon = (1 - l)/2$.

Since $|z_{n+1}/z_n|$ tends to l as $n \to \infty$, there is a positive integer N such that

$$\left|\frac{z_{n+1}}{z_n}\right| < l + \varepsilon, \qquad \text{for all } n \geq N.$$

In particular, $|z_{n+1}/z_n|$ is defined (i.e. $z_n \neq 0$) for $n \geq N$.

Thus, for $n > N$, we have

$$\left|\frac{z_n}{z_N}\right| = \left|\frac{z_n}{z_{n-1}}\right|\left|\frac{z_{n-1}}{z_{n-2}}\right| \cdots \left|\frac{z_{N+1}}{z_N}\right| < (l+\varepsilon)^{n-N},$$

for there are $n - N$ modulus terms on the right, each less than $l + \varepsilon$. Hence

$$|z_n| \leq |z_N|\,(l+\varepsilon)^{n-N}, \qquad \text{for } n \geq N. \tag{1.1}$$

Now

$$\sum_{n=N}^{\infty} |z_N|\,(l+\varepsilon)^{n-N} = |z_N| + |z_N|\,(l+\varepsilon) + |z_N|\,(l+\varepsilon)^2 + \cdots$$

is a geometric series with common ratio $l + \varepsilon$. Since $0 < l + \varepsilon < 1$, this series is convergent, and so, by Inequality (1.1) and the Comparison Test, $\sum_{n=N}^{\infty} |z_n|$ is also convergent. Hence the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent.

The inclusion of extra terms at the beginning of a series does not affect its convergence.

(b) If

$$\left|\frac{z_{n+1}}{z_n}\right| \to l \text{ as } n \to \infty, \text{ and } l > 1,$$

then there is a positive integer N such that

$$\left|\frac{z_{n+1}}{z_n}\right| > 1, \qquad \text{for all } n \geq N.$$

Put $\varepsilon = l - 1$ in the definition of convergence (or $M = 1$ in the definition of 'tends to infinity') in *Unit A3*.

(This also holds if $|z_{n+1}/z_n| \to \infty$ as $n \to \infty$.)

Thus, for $n > N$, we have

$$\left|\frac{z_n}{z_N}\right| = \left|\frac{z_n}{z_{n-1}}\right|\left|\frac{z_{n-1}}{z_{n-2}}\right|\cdots\left|\frac{z_{N+1}}{z_N}\right| > 1,$$

since each of the modulus terms is greater than 1. Hence

$$|z_n| > |z_N| > 0, \qquad \text{for } n > N,$$

and so $\{z_n\}$ cannot converge to 0 as $n \to \infty$. Thus, by the Non-null Test, the series $\sum_{n=1}^{\infty} z_n$ is divergent. ■

2 POWER SERIES

After working through this section, you should be able to:

(a) explain the terms *radius of convergence* and *disc of convergence*;

(b) use the Ratio Test to find the radius of convergence of a given series;

(c) state the Radius of Convergence Theorem;

(d) state and use the Differentiation and Integration Rules.

2.1 The radius of convergence

Earlier in the course we discussed polynomial expressions of finite degree. The theory of series enables us to go further and examine expressions that include infinitely many positive powers of z, such as

$$1 + 2z + 4z^2 + 9z^3 + \cdots.$$

Such series are known as *power* series. In the next section we shall show that any analytic function can be represented by a power series (the so-called *Taylor series* for the function); our aim here is to study power series in their own right.

A particular type of power series is the geometric series discussed in Section 1. As noted there, each value of z gives rise to a different series.

Definition Let $z \in \mathbb{C}$. An expression of the form

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots,$$

where $a_n \in \mathbb{C}, n = 0, 1, 2, \ldots$, is called a **power series about 0**.

More generally, if $\alpha \in \mathbb{C}$, then an expression of the form

$$\sum_{n=0}^{\infty} a_n (z-\alpha)^n = a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots,$$

where $a_n \in \mathbb{C}, n = 0, 1, 2, \ldots$, is called a **power series about α**.

We interpret z^0 and $(z-\alpha)^0$ in these expressions as 1, even when $z = 0$ and α respectively.

Remark Power series often appear in disguise; for example, since

$$\sum_{n=0}^{\infty} (i - 2z)^n = \sum_{n=0}^{\infty} (-1)^n 2^n (z - \tfrac{1}{2}i)^n,$$

this geometric series is a power series about $\frac{1}{2}i$.

The z in a power series is a 'variable', like the z in the rule $z \longmapsto f(z)$ for specifying a function. Different values of the complex number z in the power series $\sum_{n=0}^{\infty} a_n(z-\alpha)^n$ give different series. For some values of z the series converge; for other values they may diverge. (Theorem 2.1 below gives the details.) We say that a power series **converges on a set** S if for each $z \in S$, the corresponding series converges. For example, by Theorem 1.2, the power series

$$1 - z + z^2 - z^3 + \cdots$$

converges for $|z| < 1$ and diverges for $|z| \geq 1$. Thus, this power series converges on the open disc $D = \{z : |z| < 1\}$, and indeed on any subset of D. Furthermore, we can define a function f with domain D and rule

$$f(z) = 1 - z + z^2 - z^3 + \cdots .$$

Also, from Theorem 1.2, the power series $1 - z + z^2 - z^3 + \cdots$ converges to $1/(1+z)$ for $|z| < 1$. Thus an equivalent definition of f is

$$f(z) = 1/(1+z) \qquad (z \in D).$$

In general, if

$$A = \{z : \sum_{n=0}^{\infty} a_n(z-\alpha)^n \text{ converges}\},$$

then we can define a function f with domain A and rule

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n.$$

This accords with our usual convention that the domain of a function specified just by its rule is the set of all complex numbers to which the rule is applicable.

It is sometimes useful to refer to f as the **sum function** of the power series.

Note that in the above example we were able to give a simpler equivalent rule for f, but this is not always possible.

If we replace z by $z-1$ in the power series

$$1 - z + z^2 - z^3 + \cdots,$$

then we obtain a new power series about the point 1:

$$1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots, \qquad (*)$$

which converges to $1/z$ for $|z-1| < 1$, and diverges for $|z-1| \geq 1$. Thus the sum function of the power series $(*)$ is

$$g(z) = 1/z \qquad (z \in B),$$

where B is the open disc $\{z : |z-1| < 1\}$.

One of the remarkable features of power series is that the (sum) functions they define always turn out to have 'disc-shaped' domains. This is a consequence of the following theorem.

Theorem 2.1 Radius of Convergence Theorem

For a given power series

$$\sum_{n=0}^{\infty} a_n(z-\alpha)^n = a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots,$$

precisely one of the following possibilities occurs:

(a) the series converges only for $z = \alpha$;

(b) the series converges for all z;

(c) there is a number $R > 0$ such that

$$\sum_{n=0}^{\infty} a_n(z-\alpha)^n \text{ converges (absolutely) if } |z-\alpha| < R,$$

and

$$\sum_{n=0}^{\infty} a_n(z-\alpha)^n \text{ diverges if } |z-\alpha| > R.$$

Every power series about α converges at α (that is, for $z = \alpha$).

Remark The theorem says nothing about what happens when $|z - \alpha| = R$. We discuss this later in this section.

Case (c) of the theorem is illustrated in Figure 2.1. The series converges at all points inside the circle of radius R, centred at α, and diverges at all points outside the circle. This interpretation can be extended to case (a) by imagining a circle that has radius 0, and to case (b) by imagining a circle of infinite radius.

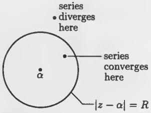


Figure 2.1

Definition The positive real number R appearing in case (c) of the Radius of Convergence Theorem is called the **radius of convergence** of the power series; we extend this definition to case (a) by writing $R = 0$, and to case (b) by writing $R = \infty$.

The use of this convenient notation does not mean that we regard ∞ as a real number.

Before proving the Radius of Convergence Theorem, we give examples to illustrate that all three of the cases mentioned in the theorem can occur.

Example 2.1

Find the radius of convergence of each of the following power series about 0.

(a) $\displaystyle\sum_{n=0}^{\infty} n^n z^n$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$ (c) $\displaystyle\sum_{n=0}^{\infty} z^n$

Solution

(a) Clearly, the power series converges for $z = 0$. But for any other value of z, the power series diverges, by the Non-null Test. Indeed, for $n > 1/|z|$, we have $|n^n z^n| = (n|z|)^n > 1$, so $\{n^n z^n\}$ is not a null sequence. Thus the series converges only for $z = 0$, and so $R = 0$.

(b) In Problem 1.11 (b), you used the Ratio Test to show that this series converges for all $z \in \mathbb{C}$. Thus $R = \infty$.

(c) This power series converges to $(1-z)^{-1}$ for $|z| < 1$, and diverges for $|z| > 1$. Thus $R = 1$. ■

Problem 2.1

(a) Find the radius of convergence of the power series $\sum_{n=0}^{\infty}(4z)^n$.

(b) More generally, find the radius of convergence of the power series $\sum_{n=0}^{\infty}(\alpha z)^n$ when $\alpha \neq 0$.

All the convergence tests in Section 1 can be applied to power series, since for each value of z a power series is just a series. The Ratio Test is particularly useful in this respect, because it can often be used to find the radius of convergence of a given power series.

Example 2.2

Find the radius of convergence of each of the following power series.

(a) $\sum_{n=0}^{\infty} n2^n(z-1)^n$ (b) $\sum_{n=0}^{\infty} \frac{(z+2i)^n}{n!}$

Solution

(a) The ratio of the $(n+1)$th and nth terms is

$$\frac{|(n+1)2^{n+1}(z-1)^{n+1}|}{|n2^n(z-1)^n|} = 2\left(1+\frac{1}{n}\right)|z-1|,$$

Since we know that the series converges for $z = 1$ we apply the test with $z \neq 1$.

which tends to $2|z-1|$ as $n \to \infty$. So, by the Ratio Test, the power series converges if $2|z-1| < 1$ and diverges if $2|z-1| > 1$. Since

$$2|z-1| < 1 \iff |z-1| < \tfrac{1}{2}$$

and

$$2|z-1| > 1 \iff |z-1| > \tfrac{1}{2},$$

the radius of convergence of the series is $1/2$.

(b) Here the ratio is

$$\left|\frac{(z+2i)^{n+1}}{(n+1)!}\right| \div \left|\frac{(z+2i)^n}{n!}\right| = \frac{n!}{(n+1)!}|z+2i| = \frac{1}{n+1}|z+2i|,$$

which tends to 0 as $n \to \infty$. So, by the Ratio Test, the power series converges for all z. Thus the radius of convergence of the series is ∞. ■

Problem 2.2

Determine the radius of convergence of each of the following power series.

(a) $\sum_{n=0}^{\infty}(2^n+4^n)z^n$ (b) $\sum_{n=0}^{\infty}\frac{(2n)!}{n!}(z+7)^n$ (c) $\sum_{n=0}^{\infty}(n+2^{-n})(z-1)^n$

It is natural to ask whether the Ratio Test can always be used to find the radius of convergence. Well, in general, we can use it to find the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n(z-\alpha)^n$ provided that

$$\frac{|a_{n+1}(z-\alpha)^{n+1}|}{|a_n(z-\alpha)^n|} = \left|\frac{a_{n+1}}{a_n}\right||z-\alpha|$$

You can check that the radius of convergence is then given by

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.$$

tends to a limit (possibly ∞) as $n \to \infty$. Sometimes this limit may not exist because the sequence $\{|a_{n+1}/a_n|\}$ may not be convergent. For example, in the following power series the ratio of successive coefficients does not converge; instead, successive ratios oscillate between 2 and $1/2$:

$$2 + z + 2z^2 + z^3 + 2z^4 + \cdots.$$

In such cases, the Ratio Test fails. Fortunately, however, the test does work for many of the power series that we shall need to consider.

We end this subsection with the proof of the Radius of Convergence Theorem.

Proof of the Radius of Convergence Theorem

This proof may be omitted on a first reading.

For simplicity let $\alpha = 0$, so that the series is of the form

$$\sum_{n=0}^{\infty} a_n z^n.$$

The proof for $\alpha \neq 0$ follows by replacing z by $z - \alpha$ in

$$\sum_{n=0}^{\infty} a_n z^n, \quad \text{for } |z| < R,$$

to obtain

$$\sum_{n=0}^{\infty} a_n (z-\alpha)^n, \quad \text{for } |z - \alpha| < R.$$

The proof depends on the claim that if a power series is convergent at some point z_0 on a circle centred at the origin, then it is *absolutely* convergent at all points z *within* the circle (see Figure 2.2). More formally, we have the following claim.

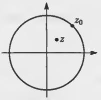


Figure 2.2

Claim If $z_0 \neq 0$ and the series $\sum_{n=0}^{\infty} a_n z_0^n$ is convergent, then the power series $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent for $|z| < |z_0|$.

To prove this claim notice that, by Theorem 1.1,

$$\text{if } \sum_{n=0}^{\infty} a_n z_0^n \text{ is convergent, then } \lim_{n\to\infty} a_n z_0^n = 0,$$

so that, for some constant K,

$$|a_n z_0^n| \leq K, \qquad n = 0, 1, 2, \ldots .$$

See *Unit A3*, Lemma 1.1.

Hence, for $z \in \mathbb{C}$,

$$|a_n z^n| = |a_n z_0^n| \left|\frac{z}{z_0}\right|^n \leq K \left|\frac{z}{z_0}\right|^n, \qquad \text{for } n = 0, 1, 2, \ldots .$$

Now, if $|z| < |z_0|$, then $|z/z_0| < 1$, and so the series

$$\sum_{n=0}^{\infty} K \left|\frac{z}{z_0}\right|^n$$

is convergent. Hence, by the Comparison Test,

$$\sum_{n=0}^{\infty} |a_n z^n|$$

is convergent, as required by the claim.

Having proved the claim, let us return to the main part of the proof.

Suppose that neither case (a) nor case (b) of the theorem holds and consider the following subset of $[0, \infty[$:

$$E = \{r \in [0, \infty[: \sum_{n=0}^{\infty} |a_n| r^n \text{ is convergent}\}.$$

Since case (a) does not hold, there is some $z_0 \neq 0$ such that $\sum_{n=0}^{\infty} a_n z_0^n$ is convergent and hence, by the claim, the set $E - \{0\}$ is non-empty. Moreover, E is an interval since, if $r_0 \in E$ and $0 < r < r_0$, then $r \in E$ (again by the claim).

Since case (b) does not hold, there is some $z_1 \neq 0$ such that $\sum_{n=0}^{\infty} a_n z_1^n$ is divergent and hence $\sum_{n=0}^{\infty} |a_n|\, |z_1|^n$ is divergent (by the Absolute Convergence Test). Thus $|z_1| \notin E$ and so the interval E has a finite, non-zero, right-hand end-point, R say, which may or may not lie in E (see Figure 2.3).

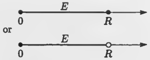


Figure 2.3

If $|z| < R$, then $\sum_{n=0}^{\infty} |a_n|\ |z|^n$ is convergent and hence $\sum_{n=0}^{\infty} a_n z^n$ is absolutely convergent, as required.

On the other hand, if $|z| > R$, then we can choose r so that $R < r < |z|$; thus $\sum_{n=0}^{\infty} |a_n| r^n$ is divergent (since $r \notin E$) and hence $\sum_{n=0}^{\infty} a_n z^n$ is divergent (by the claim).

Thus case (c) holds, and so the proof is complete. ■

2.2 The disc of convergence

According to the Radius of Convergence Theorem, if R is the radius of convergence of a power series, centred at α, then the series converges absolutely at all points of the open disc $\{z : |z - \alpha| < R\}$. This disc is called the *disc of convergence* of the series.

Definition Let R be the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n (z-\alpha)^n.$$

Then the **disc of convergence** of the power series is the open disc $\{z : |z - \alpha| < R\}$. The disc of convergence is interpreted to be $\varnothing$ if $R = 0$, and to be $\mathbb{C}$ if $R = \infty$.

Figure 2.4 illustrates the three cases.

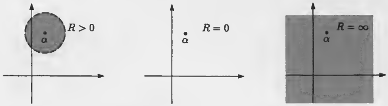


Figure 2.4 Discs of convergence

Problem 2.3

Write down the disc of convergence for each of the power series in Example 2.1 and Problem 2.2.

It is important to notice that the disc of convergence may not include *all* the points at which the power series converges. This is because the series may converge at points on the boundary of the disc. Unfortunately, the Radius of Convergence Theorem says nothing about what happens on the boundary.

In order to gain an insight into how power series can behave on the boundary of the disc of convergence, we consider three power series each with radius of convergence 1:

$$\sum_{n=0}^{\infty} z^n, \quad \sum_{n=1}^{\infty} z^n/n^2, \quad \sum_{n=1}^{\infty} z^n/n.$$

We established that the first of these power series has radius of convergence 1 in Example 2.1(c). You are now asked to establish this property for the other two.

Problem 2.4

Show that each of the following power series has radius of convergence 1.

(a) $\sum_{n=1}^{\infty} z^n/n^2$ (b) $\sum_{n=1}^{\infty} z^n/n$

Some power series converge only on their disc of convergence. For example, the power series

$$\sum_{n=0}^{\infty} z^n$$

is a geometric series, and so it diverges at every point of the boundary circle $\{z : |z| = 1\}$, by Theorem 1.2(b) (see Figure 2.5).

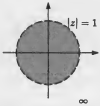


Figure 2.5 $\sum_{n=0}^{\infty} z^n$ diverges on $\{z : |z| = 1\}$

At the other extreme, there are power series which converge at every point on the boundary. For example, the power series

$$\sum_{n=1}^{\infty} z^n/n^2$$

is (absolutely) convergent at every point of the circle $\{z : |z| = 1\}$ (see Figure 2.6). This follows from the Absolute Convergence Test (Theorem 1.7), because $\sum_{n=1}^{\infty} 1/n^2$ is a convergent series.

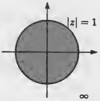


Figure 2.6 $\sum_{n=1}^{\infty} z^n/n^2$ converges on $\{z : |z| = 1\}$

Between these two extremes, it is possible for a power series to converge at some points on the boundary of the disc of convergence and diverge at others, as illustrated by the power series

$$\sum_{n=1}^{\infty} z^n/n.$$

If $z = 1$, we obtain the divergent series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots, \quad \text{(by Theorem 1.3 with } p = 1\text{);}$$

but if $z = -1$, we obtain the convergent series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots, \quad \text{(see Problem 1.10).}$$

In fact, it can be shown that this power series converges at each point of the circle $\{z : |z| = 1\}$, except 1 (see Figure 2.7).

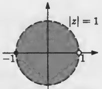


Figure 2.7 $\sum_{n=1}^{\infty} z^n/n$ diverges at 1 and converges at -1

These examples demonstrate the following observation.

> A power series may converge at none, some, or all of the points on the boundary of its disc of convergence.

You may wonder why we did not include such points in our definition of disc of convergence. We have deliberately chosen not to do so because the analytic properties of power series are best studied on regions of the complex plane, such as open discs. Our definition of the disc of convergence yields the largest region on which the power series converges.

2.3 Differentiation of power series

You have seen that the geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + z^4 + \cdots$$

defines the function

$$f(z) = \frac{1}{1-z} \qquad (|z| < 1).$$

If we differentiate this geometric series term by term, as if we were differentiating a polynomial, then we obtain a new power series:

$$\sum_{n=1}^{\infty} nz^{n-1} = 1 + 2z + 3z^2 + 4z^3 + 5z^4 + \cdots .$$

It is natural to ask whether this new series converges to the derivative

$$f'(z) = \frac{1}{(1-z)^2}.$$

The following theorem shows that it does.

Theorem 2.2 Differentiation Rule

The power series

$$\sum_{n=0}^{\infty} a_n(z-\alpha)^n \quad \text{and} \quad \sum_{n=1}^{\infty} na_n(z-\alpha)^{n-1}$$

have the same radius of convergence R, say. Furthermore, if

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n,$$

then f is analytic on the disc of convergence $\{z : |z-\alpha| < R\}$, and

$$f'(z) = \sum_{n=1}^{\infty} na_n(z-\alpha)^{n-1}, \qquad \text{for } |z-\alpha| < R.$$

Note that the series for f' does not contain an $n = 0$ term. This is analogous to the loss of the constant term when a polynomial function is differentiated.

The Differentiation Rule is an important theoretical tool that will be used in the next section to prove Taylor's Theorem. More practically, the rule can be used to find new power series from old.

Example 2.3

Use the Differentiation Rule and the geometric series

$$1 - z + z^2 - z^3 + z^4 - \cdots \qquad (*)$$

to prove that

$$\frac{1}{(1+z)^2} = 1 - 2z + 3z^2 - 4z^3 + \cdots, \qquad \text{for } |z| < 1.$$

Solution

The geometric series (*) has radius of convergence 1, and so, by the Differentiation Rule, the power series

$$-1 + 2z - 3z^2 + 4z^3 - \cdots,$$

obtained by differentiating (*) term by term, has radius of convergence 1.

Let $f(z) = 1 - z + z^2 - z^3 + z^4 - \cdots$. Then, by Theorem 1.2(a),

$$f(z) = \frac{1}{1+z} \qquad (|z| < 1)$$

and so

$$f'(z) = -\frac{1}{(1+z)^2} \qquad (|z| < 1).$$

By the Differentiation Rule,

$$f'(z) = -1 + 2z - 3z^2 + 4z^3 - \cdots, \qquad \text{for } |z| < 1.$$

Hence

$$\frac{1}{(1+z)^2} = 1 - 2z + 3z^2 - 4z^3 + \cdots, \qquad \text{for } |z| < 1. \quad \blacksquare$$

You will be given an opportunity to practise using the Differentiation Rule in Section 4. For now, let us turn our attention to the proof of the rule.

Proof Our proof is in two steps. Again, for simplicity, we take $\alpha = 0$.

This proof may be omitted on a first reading.

(a) Let the power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} n a_n z^{n-1}$ have radii of convergence R and R', respectively. We shall prove that $R = R'$.

We prove that $R \le R'$ by showing that if $|z| < R$, then the power series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges. To prove this, let r be a real number such that $|z| < r < R$. Then the series $\sum_{n=0}^{\infty} a_n r^n$ converges, and so $\lim_{n \to \infty} a_n r^n = 0$. Thus there is a number K such that

$$|a_n r^n| \le K, \qquad \text{for } n = 1, 2, \ldots.$$

See *Unit A3*, Lemma 1.1.

We now write

$$\begin{aligned} |n a_n z^{n-1}| &= n |a_n r^n| \left|\frac{z}{r}\right|^{n-1} r^{-1} \\ &\le n r^{-1} K \left|\frac{z}{r}\right|^{n-1}, \qquad \text{for } n = 1, 2, \ldots. \end{aligned}$$

Since $|z| < r$, it follows that $|z/r| < 1$, so the series $\sum_{n=1}^{\infty} n r^{-1} K \left|\frac{z}{r}\right|^{n-1}$ converges, by the Ratio Test. Hence, by the Comparison Test, the power series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges absolutely, as required. Thus $R \le R'$.

$$\lim_{n \to \infty} \frac{(n+1)|z/r|^n}{n|z/r|^{n-1}} = \left|\frac{z}{r}\right| < 1.$$

We now prove that $R \ge R'$ by showing that, if $|z| < R'$, then the power series $\sum_{n=1}^{\infty} a_n z^n$ converges. Since $|z| < R'$, $\sum_{n=1}^{\infty} |n a_n z^{n-1}|$ is convergent and, by the Multiple Rule, so is $\sum_{n=1}^{\infty} |n a_n z^n|$. But

We just multiply by $|z|$.

$$|a_n z^n| \le |n a_n z^n|, \qquad \text{for } n = 1, 2, \ldots.$$

So, by the Comparison Test, the power series $\sum_{n=1}^{\infty} a_n z^n$ is absolutely convergent, as required. Thus $R \ge R'$.

It follows that $R = R'$: the two radii of convergence are equal.

(b) Next we show that f is analytic on $D = \{z : |z| < R\}$ and $f'(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$, for $|z| < R$. To do this, let z and z_0 be arbitrary distinct points in the disc D (see Figure 2.8). Then, by the Combination Rules for series,

$$\begin{aligned}
&\left|\frac{f(z)-f(z_0)}{z-z_0} - \sum_{n=1}^{\infty} na_n z_0^{n-1}\right| \\
&= \left|\sum_{n=1}^{\infty} a_n\left(\frac{z^n - z_0^n}{z-z_0} - nz_0^{n-1}\right)\right| \\
&= \left|\sum_{n=1}^{\infty} a_n(z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1} - nz_0^{n-1})\right| \\
&\le |p_N(z)| + |q_N(z)|, \text{ say,}
\end{aligned}$$

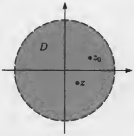


Figure 2.8

where, for each N, p_N is the polynomial function

$$p_N(z) = \sum_{n=1}^{N} a_n(z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1} - nz_0^{n-1}),$$

and

$$q_N(z) = \sum_{n=N+1}^{\infty} a_n(z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1} - nz_0^{n-1}).$$

Here we are using the result that $z^n - z_0^n$ is equal to
$(z - z_0)(z^{n-1} + \cdots + z_0^{n-1})$;
see *Unit A1*, Theorem 1.3(b).

To prove that $f'(z_0) = \sum_{n=1}^{\infty} na_n z_0^{n-1}$, it is sufficient to show that, for each $\varepsilon > 0$, there is a $\delta > 0$ and an integer N such that

$$|z - z_0| < \delta \implies |p_N(z)| < \varepsilon/2 \text{ and } |q_N(z)| < \varepsilon/2.$$

To ensure that $|q_N(z)| < \varepsilon/2$, let r be a real number between $|z_0|$ and R (see Figure 2.9), and consider the series $\sum_{n=1}^{\infty} na_n r^{n-1}$. Since this series is absolutely convergent, we can choose N so that $\sum_{n=N+1}^{\infty} n|a_n|r^{n-1} < \varepsilon/4$.

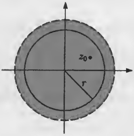


Figure 2.9 $|z_0| < r < R$

Now take $\delta = r - |z_0|$. If $|z - z_0| < \delta$ (see Figure 2.10), then

$$|z| \le |z - z_0| + |z_0| < \delta + |z_0| = (r - |z_0|) + |z_0| = r.$$

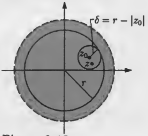


Figure 2.10

So $|z| < r$. Furthermore, since $|z_0| < r$, we have

$$\begin{aligned}
|q_N(z)| &= \left|\sum_{n=N+1}^{\infty} a_n(z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1} - nz_0^{n-1})\right| \\
&\le \sum_{n=N+1}^{\infty} 2n|a_n|r^{n-1} = 2\sum_{n=N+1}^{\infty} n|a_n|r^{n-1} \\
&< 2(\varepsilon/4) = \varepsilon/2 \quad \text{(by our choice of } N).
\end{aligned}$$

Finally, to ensure that $|p_N(z)| < \varepsilon/2$, notice that p_N is a polynomial function such that $p_N(z_0) = 0$. Since polynomial functions are continuous, we can, if necessary, make δ even smaller than $r - |z_0|$, so that

$$|p_N(z)| = |p_N(z) - p_N(z_0)| < \varepsilon/2, \qquad \text{for } |z - z_0| < \delta.$$

This completes the proof that $f'(z_0) = \sum_{n=1}^{\infty} na_n z_0^{n-1}$ because, given any $\varepsilon > 0$, we have shown how to find δ and N so that $|z - z_0| < \delta$ implies

$$\begin{aligned}
\left|\frac{f(z)-f(z_0)}{z-z_0} - \sum_{n=1}^{\infty} na_n z_0^{n-1}\right| &\le |p_N(z)| + |q_N(z)| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \blacksquare
\end{aligned}$$

The following corollary shows that any power series can be integrated term by term on its disc of convergence.

Corollary Integration Rule

The power series

$$\sum_{n=0}^{\infty} a_n(z-\alpha)^n \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-\alpha)^{n+1}$$

have the same radius of convergence R, say. Furthermore, if $f(z)=\sum_{n=0}^{\infty} a_n(z-\alpha)^n$, then the function

$$F(z) = \text{ constant } + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-\alpha)^{n+1}$$

is a primitive of f on $\{z : |z-\alpha| < R\}$.

Proof In order to see how this corollary follows from the Differentiation Rule, we denote the constant term in F by b_0, and let

$$b_1 = a_0,\ b_2 = \frac{a_1}{2},\ b_3 = \frac{a_2}{3}, \ldots,\ b_n = \frac{a_{n-1}}{n}, \ldots.$$

Then we can write

$$F(z) = \sum_{n=0}^{\infty} b_n(z-\alpha)^n \quad \text{and} \quad f(z) = \sum_{n=1}^{\infty} nb_n(z-\alpha)^{n-1}.$$

It is now clear from the Differentiation Rule that the two series have the same radius of convergence R, that F is analytic on $\{z : |z-\alpha| < R\}$, and that $F'(z) = \sum_{n=1}^{\infty} nb_n(z-\alpha)^{n-1} = f(z)$, for $|z-\alpha| < R$. In other words, F is a primitive of f on $\{z : |z-\alpha| < R\}$. ■

Example 2.4

Use the Integration Rule and the geometric series

$$1 - z + z^2 - z^3 + z^4 - \cdots \qquad (*)$$

We return to this example at the beginning of the next section.

to prove that

$$\operatorname{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \qquad \text{for } |z| < 1.$$

Solution

The geometric series $(*)$ has radius of convergence 1 and so, by the Integration Rule, the power series

$$z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots,$$

obtained by integrating $(*)$ term by term, has radius of convergence 1.

Let $f(z) = 1 - z + z^2 - z^3 + z^4 - \cdots$. Then, by the Integration Rule, the function

$$F(z) = \text{ constant } + z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

is a primitive of f on $\{z : |z| < 1\}$. But $f(z) = 1/(1+z)$ $(|z| < 1)$, and so

$$z \longmapsto \text{ constant } + \operatorname{Log}(1+z)$$

is also a primitive of f on $\{z : |z| < 1\}$. Since $\operatorname{Log} 1 = 0$, by comparing the two forms for the primitive of f, we deduce that

$$\operatorname{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \qquad \text{for } |z| < 1. \quad ■$$

3 TAYLOR'S THEOREM

After working through this section, you should be able to:

(a) state Taylor's Theorem;

(b) find the Taylor series about a point α for a function f;

(c) show that the Taylor series for a function f converges to $f(z)$ for z in some suitably chosen disc.

3.1 Taylor series

In the example at the end of the previous section you saw that

$$\operatorname{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \qquad \text{for } |z| < 1.$$

We say that the function

$$F(z) = \operatorname{Log}(1+z),$$

which has domain $\mathbb{C} - \{x \in \mathbb{R} : x \le -1\}$, is 'represented' on the open disc $\{z : |z| < 1\}$ by the power series

$$z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots;$$

(see Figure 3.1).

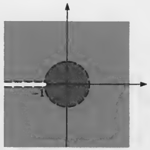


Figure 3.1 The domain of F and the disc on which F is represented by the power series

By the Differentiation Rule, any function that can be represented by a power series on an open disc D in this way must be analytic on D. But what is more remarkable is the fact that this process can be reversed. Any function that is analytic on an open disc D can be represented by a power series on D that is 'Taylor-made' for the purpose.

> **Theorem 3.1 Taylor's Theorem**
>
> If f is a function which is analytic on the open disc $D = \{z : |z - \alpha| < r\}$, then
>
> $$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n, \qquad \text{for } z \in D. \tag{3.1}$$
>
> Moreover, this representation of f is unique, in the sense that if
>
> $$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n, \qquad \text{for } z \in D,$$
>
> then $a_n = f^{(n)}(\alpha)/n!$, for $n = 0, 1, 2, \ldots$.

Brook Taylor (1685–1731) studied in Cambridge. Between the years 1712 and 1719, he wrote numerous papers on such subjects as oscillations, capillarity and projectiles. Taylor's Theorem (for real functions) was discovered in 1715, but his proof contains no discussion of convergence, and would certainly not be accepted today. The complex version of Taylor's Theorem is due to Cauchy.

Remarks

1 Note that Equation (3.1) asserts the *equality* of the value of the given function f and of the power series for each value of z in D. This is in contrast to Section 2 where, in writing

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n,$$

we were using the series to *define* the function f.

2 The uniqueness of the power series representation of a function about a given point is an important result, which we shall use often.

3 Taylor's Theorem shows that a function f which is analytic at α can be represented as a power series about α, whose coefficients are of the form $f^{(n)}(\alpha)/n!$. Of course, this can be the case only if f has derivatives of all orders at α, a result which was stated (but not proved) in *Unit B2* as part of Cauchy's nth Derivative Formula. We shall obtain a proof of Cauchy's nth Derivative Formula from the proof of Taylor's Theorem.

4 The term $f^{(n)}(\alpha)/n!$ makes sense for $n = 0$ because, by convention, we take $0! = 1$ and $f^{(0)}(z) = f(z)$.

5 Taylor's Theorem is a powerful result that enables us to use power series to investigate the properties of analytic functions. You will see several examples of this later in the unit, and many more throughout the course.

Any series of the form in Equation (3.1) is called a *Taylor Series.*

Definition If f is a function with derivatives $f'(\alpha), f''(\alpha), f'''(\alpha), \ldots$ at the point α, then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n$$

is called the **Taylor series about α for f**. The coefficient $f^{(n)}(\alpha)/n!$ is known as the **nth Taylor coefficient (of f at α)**.

Sometimes it is possible to find the Taylor series for a function f directly from the definition.

Example 3.1

Find the Taylor series about 0 for the function $f(z) = e^z$.

Solution

The Taylor series is $\sum_{n=0}^{\infty} a_n z^n$ where $a_n = f^{(n)}(0)/n!$.

Since the exponential function is its own derivative, all the higher derivatives $f^{(n)}(z)$ are equal to e^z, and so

$$f^{(n)}(0) = e^0 = 1.$$

The Taylor series about 0 for the exponential function is therefore

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots. \quad \blacksquare$$

The Taylor series about α for a function f is of little use unless we know that there are values of z around α for which the series converges to $f(z)$. Taylor's Theorem ensures that this is the case for all z in any open disc D, centred at α, on which f is analytic.

Since the function $f(z) = e^z$ in Example 3.1 is entire, it must be analytic on every open disc centred at 0. So the series in the example must converge to e^z for all z in $\mathbb{C}$. More generally we have the following corollary to Taylor's Theorem.

Corollary If f is an entire function, then the Taylor series about any point α for f converges to $f(z)$ for all $z \in \mathbb{C}$.

Example 3.2

Find the Taylor series about 0 for the function $f(z) = \cos z$. Explain why the series converges to $\cos z$ for all $z \in \mathbb{C}$.

Solution

Since the function $f(z) = \cos z$ is entire, it follows from the corollary that its Taylor series about 0 must converge to $\cos z$ for all $z \in \mathbb{C}$. The Taylor series is found by calculating the higher derivatives of f at 0:

$$\begin{aligned}
f(z) &= \cos z, && \text{so} \quad f(0) = 1;\\
f'(z) &= -\sin z, && \text{so} \quad f'(0) = 0;\\
f^{(2)}(z) &= -\cos z, && \text{so} \quad f^{(2)}(0) = -1;\\
f^{(3)}(z) &= \sin z, && \text{so} \quad f^{(3)}(0) = 0;\\
f^{(4)}(z) &= \cos z, && \text{so} \quad f^{(4)}(0) = 1.
\end{aligned}$$

Since every fourth differentiation brings us back to $f(z)$, the above pattern repeats itself. The Taylor series about 0 for the function $f(z) = \cos z$ is therefore

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C}. \quad \blacksquare$$

Problem 3.1

For each of the following functions f, find the Taylor series about 0 for f, and explain why it converges to $f(z)$ for all $z \in \mathbb{C}$.

(a) $f(z) = \sin z$ (b) $f(z) = \cosh z$ (c) $f(z) = \sinh z$

If f is not entire, then it may not be possible to find a Taylor series that converges to $f(z)$ for all z in the domain of f. In such cases, it often helps to identify a region $\mathcal{R}$ on which f is analytic. We can then pick a point α in $\mathcal{R}$ and try to find the Taylor series about α for f. By Taylor's Theorem, this series converges to $f(z)$ for all z in any open disc D, centred at α, that lies within the region $\mathcal{R}$ (see Figure 3.2). We often choose D to be as large as possible (see Figure 3.3).

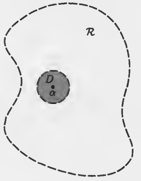


Figure 3.2

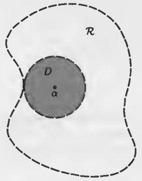


Figure 3.3 The 'largest' open disc D

Example 3.3

Find the Taylor series about 0 for the function $f(z) = \text{Log}(1 + z)$. Show that the series converges to $\text{Log}(1 + z)$ for $|z| < 1$ and check that the series is the same as the one given in Example 2.4.

Solution

The function f is analytic on the region $\mathbb{C} - \{x \in \mathbb{R} : x \leq -1\}$. The largest open disc, centred at 0, that will fit in this region is $D = \{z : |z| < 1\}$ (see Figure 3.4). So, by Taylor's Theorem, the Taylor series about 0 for f converges to $f(z)$, for $|z| < 1$.

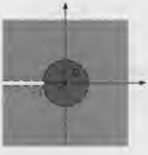


Figure 3.4

The Taylor series is found by calculating the higher derivatives of f at 0:

$$\begin{aligned}
&f(z) = \mathrm{Log}(1+z), && \text{so} \quad f(0) = 0;\\
&f'(z) = (1+z)^{-1}, && \text{so} \quad f'(0) = 1;\\
&f^{(2)}(z) = -(1+z)^{-2}, && \text{so} \quad f^{(2)}(0) = -1;\\
&f^{(3)}(z) = 2(1+z)^{-3}, && \text{so} \quad f^{(3)}(0) = 2;\\
&f^{(4)}(z) = (-3)(2)(1+z)^{-4}, && \text{so} \quad f^{(4)}(0) = -3!.
\end{aligned}$$

In general,

$$f^{(n)}(z) = \frac{(-1)^{n+1}(n-1)!}{(1+z)^n}, \text{so } \frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n+1}}{n}, \quad \text{for } n = 1, 2, \ldots.$$

This could be established formally by Mathematical Induction.

Thus the Taylor series about 0 for f is

$$\mathrm{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots \qquad \text{for } |z| < 1,$$

which is the same as the series given in Example 2.4. ■

Of course, it is no coincidence that the power series are the same because Taylor's Theorem states that the power series representation of a function about a given point is unique. It follows that the series must be the same.

Problem 3.2

Find the Taylor series about 0 for the function $f(z) = (1+z)^{-3}$. Show that the series converges to $(1+z)^{-3}$ for $|z| < 1$.

So far, all the Taylor series we have found have been about the point 0, but Taylor series about other points can be found in a similar way.

Example 3.4

Find the Taylor series about 2 for $f(z) = \mathrm{Log}(1+z)$. Show that the series converges to $\mathrm{Log}(1+z)$ for $|z-2| < 3$.

Solution

The function f is analytic on the region $\mathbb{C} - \{x \in \mathbb{R} : x \leq -1\}$. The largest open disc, centred at 2, that will fit in this region is $D = \{z : |z-2| < 3\}$ (see Figure 3.5). So, by Taylor's Theorem, the Taylor series about 2 for f converges to $f(z)$ for $|z-2| < 3$.

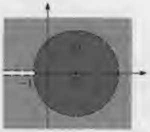


Figure 3.5
$D = \{z : |z-2| < 3\}$

The Taylor series is found by calculating the higher derivatives of f at 2:

$$\begin{aligned}
&f(z) = \mathrm{Log}(1+z), && \text{so} \quad f(2) = \mathrm{Log}(3) = \log_e 3;\\
&f'(z) = (1+z)^{-1}, && \text{so} \quad f'(2) = 1/3;\\
&f^{(2)}(z) = -(1+z)^{-2}, && \text{so} \quad f^{(2)}(2) = -1/3^2;\\
&f^{(3)}(z) = 2(1+z)^{-3}, && \text{so} \quad f^{(3)}(2) = 2/3^3;\\
&f^{(4)}(z) = (-3)(2)(1+z)^{-4}, && \text{so} \quad f^{(4)}(2) = -3!/3^4.
\end{aligned}$$

In general,

$$f^{(n)}(z) = \frac{(-1)^{n+1}(n-1)!}{(1+z)^n}, \quad \text{so} \quad \frac{f^{(n)}(2)}{n!} = \frac{(-1)^{n+1}}{n\,3^n}, \text{ for } n = 1, 2, \ldots.$$

Thus the Taylor series about 2 for f is

$$\mathrm{Log}(1+z) = \log_e 3 + \frac{(z-2)}{3} - \frac{(z-2)^2}{2 \times 3^2} + \frac{(z-2)^3}{3 \times 3^3} - \frac{(z-2)^4}{4 \times 3^4} + \cdots,$$
$$\text{for } |z-2| < 3. \quad ■$$

Warning: Note that we do not expand brackets of the form $(z-2)^n$ because this is a power series about 2.

Sometimes it is clearer to indicate how a series continues by giving an expression for the general term. For example, above we could have written

$$\mathrm{Log}(1+z) = \log_e 3 + \frac{(z-2)}{3} - \frac{(z-2)^2}{2 \times 3^2} + \cdots + \frac{(-1)^{n+1}(z-2)^n}{n3^n} + \cdots,$$

or, equivalently,

$$\mathrm{Log}(1+z) = \log_e 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(z-2)^n}{n3^n}.$$

Problem 3.3

Find the Taylor series about i for the function $f(z) = 1/z$, giving an expression for the general term.

To end this subsection we ask you to use the uniqueness property of Taylor series to prove an interesting result about the Taylor series of odd and even functions. A function $f:A \longrightarrow \mathbb{C}$ is **even** if

$$f(-z) = f(z), \qquad \text{for } z \in A,$$

whereas f is **odd** if

$$f(-z) = -f(z), \qquad \text{for } z \in A.$$

For example, cos is even, whereas sin is odd.

Problem 3.4

Let the function f be analytic at 0, with Taylor series about 0:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Prove that:

(a) if f is even, then $a_n = 0$ for n odd;

(b) if f is odd, then $a_n = 0$ for n even.

3.2 Basic Taylor series

In theory, we could use Taylor's Theorem to calculate Taylor series for any given function. However, in practice, it is not always easy to find the higher derivatives of the function concerned. In the next section, we illustrate how it is often easier to calculate a Taylor series by applying the rules for series (discussed in Sections 1 and 2) to a list of basic functions whose Taylor series are already known. Some frequently occurring Taylor series (about 0) that can be used for this purpose are listed below.

Basic Taylor series

$$(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots, \qquad \text{for } |z| < 1;$$

$$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots, \qquad \text{for } z \in \mathbb{C};$$

$$\operatorname{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \qquad \text{for } |z| < 1;$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \qquad \text{for } z \in \mathbb{C};$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \qquad \text{for } z \in \mathbb{C};$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots, \qquad \text{for } z \in \mathbb{C};$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots, \qquad \text{for } z \in \mathbb{C}.$$

In addition to these basic Taylor series, we frequently use Taylor series about 0 for functions of the form $f(z) = (1+z)^\alpha$, where α is a complex number. Such Taylor series are known as **binomial series**.

You have already investigated one binomial series in Problem 3.2, where you found that

$$(1+z)^{-3} = 1 - \frac{(3)(2)}{2}z + \frac{(4)(3)}{2}z^2 - \frac{(5)(4)}{2}z^3 + \cdots, \qquad \text{for } |z| < 1.$$

More generally, we have the following result.

Theorem 3.2 Binomial Series

If $\alpha \in \mathbb{C}$, then the binomial series about 0 for the function $f(z) = (1+z)^\alpha$ is

$$(1+z)^\alpha = 1 + \binom{\alpha}{1}z + \binom{\alpha}{2}z^2 + \binom{\alpha}{3}z^3 + \cdots, \qquad \text{for } |z| < 1,$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-(n-1))}{n!}.$$

If α is a non-negative integer, then the binomial series reduces to a polynomial; otherwise, the series is a power series whose radius of convergence is 1.

This is a generalization of the Binomial Theorem given in *Unit A1*. There $\alpha \in \mathbb{N}$. The generalization to $\alpha \in \mathbb{R}$ is due to Isaac Newton, and to $\alpha \in \mathbb{C}$ is due to Cauchy.

Proof If $f(z) = (1+z)^\alpha$, then

$$\begin{aligned}
f(z) &= (1+z)^\alpha, && \text{so} \quad f(0) = 1;\\
f'(z) &= \alpha(1+z)^{\alpha-1}, && \text{so} \quad f'(0) = \alpha;\\
f^{(2)}(z) &= \alpha(\alpha-1)(1+z)^{\alpha-2}, && \text{so} \quad f^{(2)}(0) = \alpha(\alpha-1);\\
f^{(3)}(z) &= \alpha(\alpha-1)(\alpha-2)(1+z)^{\alpha-3}, && \text{so} \quad f^{(3)}(0) = \alpha(\alpha-1)(\alpha-2).
\end{aligned}$$

If α is a non-negative integer then all derivatives beyond the αth, and hence all Taylor coefficients beyond the αth, are zero.

In general, the nth Taylor coefficient is

$$\frac{f^{(n)}(0)}{n!} = \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!} = \binom{\alpha}{n}.$$

The function $f(z) = (1+z)^\alpha = \exp(\alpha \operatorname{Log}(1+z))$ is analytic on the open disc $D = \{z : |z| < 1\}$, so, by Taylor's Theorem, the binomial series converges to $(1+z)^\alpha$ for $|z| < 1$. So the radius of convergence is at least 1.

To prove that the radius of convergence of the binomial series is equal to 1 we apply the Ratio Test. The ratio of successive terms is given by

$$\begin{aligned}
\left|\binom{\alpha}{n+1}z^{n+1}\right| \div \left|\binom{\alpha}{n}z^n\right| &= \left|\frac{\alpha(\alpha-1)\dots(\alpha-n)n!}{\alpha(\alpha-1)\dots(\alpha-(n-1))(n+1)!}\right||z|\\
&= \left|\frac{\alpha-n}{n+1}\right||z|.
\end{aligned}$$

This converges to $|z|$ as $n \to \infty$, since

$$|(\alpha-n)/(n+1)| = |((\alpha/n)-1)/(1+1/n)| \to 1 \text{ as } n \to \infty.$$

The binomial series therefore converges for $|z| < 1$ and diverges for $|z| > 1$; so the radius of convergence of the series is 1, as shown in Figure 3.6. ■

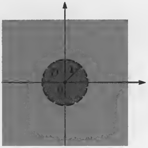


Figure 3.6

3.3 Proof of Taylor's Theorem and Cauchy's nth Derivative Formula

To end this section we prove Taylor's Theorem and then use the proof to give a proof of Cauchy's nth Derivative Formula that was introduced, without proof, in *Unit B2*.

This subsection may be omitted on a first reading.

Theorem 3.1 Taylor's Theorem

If f is a function which is analytic on an open disc $D = \{z : |z - \alpha| < r\}$, then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n, \qquad \text{for } z \in D.$$

Moreover, this representation of f is unique, in the sense that if

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n, \qquad \text{for } z \in D,$$

then $a_n = f^{(n)}(\alpha)/n!$.

Proof There are six steps in the proof. We first (a) represent f as an integral, then (b) approximate the integrand by a polynomial in $z - \alpha$, then (c) integrate the polynomial term by term, and then (d) obtain a power series representation of f by letting $n \to \infty$. We complete the proof by differentiating the series n times; this shows (e) that the coefficients of the series are $f^{(n)}(\alpha)/n!$ and (f) that the coefficients are unique.

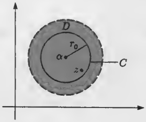


Figure 3.7

(a) Let z be an arbitrary point in D. By choosing r_0 so that $|z - \alpha| < r_0 < r$, we can ensure that the circle C with centre α and radius r_0 lies in D, and encloses the point z (see Figure 3.7). By applying Cauchy's Integral Formula to the function f, we can express $f(z)$ in terms of the values of f on the circle C:

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw.$$

For Cauchy's Integral Formula see *Unit B2*, Theorem 2.1.

(b) Next, we note that

$$\frac{1}{w-z} = \frac{1}{(w-\alpha)-(z-\alpha)} = \frac{1}{w-\alpha}\left(1 - \frac{z-\alpha}{w-\alpha}\right)^{-1}. \qquad (*)$$

But, for any $\lambda \in \mathbb{C}, n \in \mathbb{N}$, we have

$$(1-\lambda)^{-1} = 1 + \lambda + \lambda^2 + \cdots + \lambda^{n-1} + \frac{\lambda^n}{1-\lambda},$$

as you can check by multiplying both sides of the equation by $1 - \lambda$.

Replacing λ by $(z-\alpha)/(w-\alpha)$, we obtain

$$\left(1 - \frac{z-\alpha}{w-\alpha}\right)^{-1} = 1 + \frac{z-\alpha}{w-\alpha} + \cdots + \frac{(z-\alpha)^{n-1}}{(w-\alpha)^{n-1}} + \frac{((z-\alpha)/(w-\alpha))^n}{1-(z-\alpha)/(w-\alpha)}.$$

So, from $(*)$, we have

$$\frac{1}{w-z} = \frac{1}{w-\alpha} + \frac{z-\alpha}{(w-\alpha)^2} + \cdots + \frac{(z-\alpha)^{n-1}}{(w-\alpha)^n} + \frac{((z-\alpha)/(w-\alpha))^n}{(w-\alpha)-(z-\alpha)}.$$

(c) Substituting the above expression for $1/(w - z)$ into the integral for $f(z)$, we obtain

$$\begin{aligned} f(z) = {} & \frac{1}{2\pi i}\int_C \frac{f(w)}{w-\alpha}\,dw + \frac{z-\alpha}{2\pi i}\int_C \frac{f(w)}{(w-\alpha)^2}\,dw + \cdots \\ & + \frac{(z-\alpha)^{n-1}}{2\pi i}\int_C \frac{f(w)}{(w-\alpha)^n}\,dw + \frac{(z-\alpha)^n}{2\pi i}\int_C \frac{f(w)}{(w-\alpha)^n(w-z)}\,dw. \end{aligned}$$

We now write this in the form

$$f(z) = b_0 + b_1(z-\alpha) + b_2(z-\alpha)^2 + \cdots + b_{n-1}(z-\alpha)^{n-1} + I_n,$$

where

$$b_m = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-\alpha)^{m+1}}\,dw, \qquad \text{for } m = 0, 1, \ldots, \qquad (\dagger)$$

and

$$I_n = \frac{(z-\alpha)^n}{2\pi i}\int_C \frac{f(w)}{(w-\alpha)^n(w-z)}\,dw.$$

(d) Next, we use the Estimation Theorem, *Unit B1*, Theorem 4.1, to show that $I_n \to 0$ as $n \to \infty$. Since f is analytic on D, f is continuous on C, and thus bounded on C; that is, there is a number M such that $|f(w)| \le M$ for $w \in C$. Also, for $w \in C$, we have

C is compact so this follows from the Boundedness Theorem, *Unit A3*, Theorem 5.3.

$$|w - z| \ge |w-\alpha| - |z-\alpha| = r_0 - |z-\alpha|,$$

and so it follows from the Estimation Theorem that

$$\begin{aligned}|I_n| &\le \frac{|z-\alpha|^n}{2\pi} \times \frac{M}{r_0^n(r_0 - |z-\alpha|)} \times 2\pi r_0\\ &= \frac{Mr_0}{r_0 - |z-\alpha|}\left|\frac{z-\alpha}{r_0}\right|^n.\end{aligned}$$

But $|z-\alpha| < r_0$, and so the right-hand side tends to 0 as $n \to \infty$. Hence I_n tends to 0 as $n \to \infty$. Since z is an arbitrary point of D it follows that

Squeeze Rule, *Unit A3*, Theorem 1.1

$$f(z) = \sum_{n=0}^{\infty} b_n(z-\alpha)^n, \qquad \text{for } z \in D,$$

where $b_0, b_1, b_2, \ldots,$ are given by $(\dagger)$.

(e) By the Differentiation Rule, f must have derivatives of all orders at α, and the coefficients b_n must be equal to $f^{(n)}(\alpha)/n!$. Indeed,

$$\begin{aligned}
f(z) &= b_0 + b_1(z-\alpha) + b_2(z-\alpha)^2 + \cdots,\\
f'(z) &= 1b_1 + 2b_2(z-\alpha) + 3b_3(z-\alpha)^2 + \cdots,\\
f''(z) &= 2\cdot 1b_2 + 3\cdot 2b_3(z-\alpha) + 4\cdot 3b_4(z-\alpha)^2 + \cdots,\\
f^{(3)}(z) &= 3\cdot 2\cdot 1b_3 + 4\cdot 3\cdot 2b_4(z-\alpha) + 5\cdot 4\cdot 3b_5(z-\alpha)^2 + \cdots,\\
&\ \ \vdots
\end{aligned}$$

It follows that $f'(\alpha) = 1!b_1$, $f''(\alpha) = 2!b_2$, $f^{(3)}(\alpha) = 3!b_3$, and, in general, $f^{(n)}(\alpha) = n!b_n$, for $n = 1, 2, \ldots,$ as required.

This could be established formally by Mathematical Induction.

(f) Finally, notice that the same differentiation argument shows that the representation is unique. For if

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n, \qquad \text{for } z \in D,$$

then $a_n = f^{(n)}(\alpha)/n! = b_n$, for $n = 1, 2, \ldots.$ ■

With remarkably little effort we can now prove Cauchy's nth Derivative Formula.

Cauchy's nth Derivative Formula

Let $\mathcal{R}$ be a simply-connected region, let Γ be a simple-closed contour in $\mathcal{R}$, and let f be a function which is analytic on $\mathcal{R}$. Then, for any point α inside Γ, f is n-times differentiable at α and

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z-\alpha)^{n+1}}\,dz, \qquad \text{for } n = 1, 2, \ldots .$$

Unit B2, Theorem 3.2

Proof Let $D = \{z : |z-\alpha| < r\}$ be an open disc inside Γ centred at α. Since f is analytic on D we can proceed to use the same arguments as in the proof of Taylor's Theorem. In particular we can choose a circle $C = \{w : |w-\alpha| = r_0\}$ in D centred at α and show that the nth Taylor coefficient of f at α is

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-\alpha)^{n+1}}\,dw, \qquad \text{for } n = 1, 2 \ldots .$$

But from part (e) of the proof of Taylor's Theorem we know that f is n-times differentiable at α and $f^{(n)}(\alpha) = n!b_n$, so

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-\alpha)^{n+1}}\,dw, \qquad \text{for } n = 1, 2, \ldots .$$

Since $\mathcal{R}$ is simply-connected, and C lies inside Γ, we can deduce Cauchy's nth Derivative Formula by applying the Shrinking Contour Theorem. ■

Unit B2, Theorem 1.4

4 MANIPULATING TAYLOR SERIES

After working through this section, you should be able to:

(a) use the rules for manipulating power series to find the Taylor series for a given function.

4.1 Finding Taylor series (audio-tape)

In the previous section, we used the formula for Taylor coefficients to find a list of basic Taylor series. We were able to do this because it was relatively easy to calculate the higher derivatives of the functions involved. Unfortunately, for many functions the calculation of the higher derivatives can be quite messy, which makes the formula difficult to apply.

In this audio tape, we illustrate how to find the Taylor series for many functions by applying the rules for manipulating power series to the basic Taylor series found in the previous section. (Note that in the audio tape we are principally concerned to find the coefficients of the Taylor series. We return to the question of radius of convergence after the tape.)

Several of the rules that are needed for this purpose were introduced earlier in the unit, where we showed that we can add series, take multiples of series, and differentiate and integrate power series as if they were polynomials. In the tape, this similarity with polynomials is further reinforced when we introduce two more rules which show that power series can be multiplied and composed as if they were polynomials.

In the course of this tape we shall determine the Taylor series about 0 for the inverse functions $\tan^{-1}$ and $\sin^{-1}$. As yet we have not given a definition of these inverse functions. To do this it is necessary to restrict the domains of the functions tan and sin in such a way that the restricted functions are one-one, and hence have inverse functions. In *Unit C2* we will show that a convenient way to do this is to restrict the domains of both tan and sin to the region $S = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$.

It turns out that both tan and sin are one-one on S and that the images $\tan(S)$ and $\sin(S)$ are as shown in Figures 4.1 and 4.2, respectively. So, by restricting the domains of tan and sin to S we are able to define the inverse functions $\tan^{-1}$ and $\sin^{-1}$ with the domains shown in the figures.

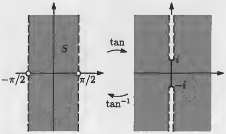


Figure 4.1

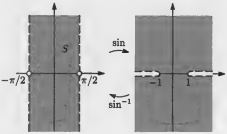


Figure 4.2

To find the Taylor series about 0 for the functions $\tan^{-1}$ and $\sin^{-1}$ we need to know that these functions are analytic, and we need to be able to find their derivatives. The analyticity of these functions is established later in the course. Their derivatives can be obtained by using the Inverse Function Rule given in *Unit A4*. The results are the same as for the corresponding real inverse functions:

$$(\tan^{-1})'(z) = 1/(1+z^2) \quad \text{and} \quad (\sin^{-1})'(z) = 1/\sqrt{1-z^2}.$$

NOW START THE TAPE.

1. Some basic Taylor series about 0

geometric series

$$(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots, \quad \text{for } |z|<1;$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \quad \text{for } z \in \mathbb{C};$$

$$\mathrm{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \quad \text{for } |z|<1;$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \text{for } z \in \mathbb{C};$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C};$$

binomial series

$$(1+z)^\alpha = 1 + \binom{\alpha}{1} z + \binom{\alpha}{2} z^2 + \cdots, \quad \text{for } |z|<1,$$

where $\binom{\alpha}{n} = \dfrac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))}{n!}$.

nth binomial coefficient

For example:

$$(1+z)^{-2} = 1 + \binom{-2}{1} z + \binom{-2}{2} z^2 + \binom{-2}{3} z^3 + \cdots, \quad \text{for } |z|<1,$$

where

$$\binom{-2}{1} = -2, \quad \binom{-2}{2} = \frac{(-2)(-3)}{2!} = 3, \quad \binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4,$$

so

$$(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \cdots, \quad \text{for } |z|<1.$$

$(1+z)^\alpha$ yields a polynomial if α is a non-negative integer.

2. Problem 4.1

Find the Taylor series about 0 for $h(z) = (1+z)^{-1/2}$.

3. Taylor series about 0 for $h(z) = 2(1-z)^{-1} - e^z$

$$(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots, \quad \text{for } |z|<1;$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

Section 1

Combination Rules Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n, \quad \text{for } |z-\alpha| < R,$$

$$g(z) = \sum_{n=0}^{\infty} b_n (z-\alpha)^n, \quad \text{for } |z-\alpha| < R'.$$

Taylor series about α

Sum Rule If $r = \min\{R, R'\}$, then

$$(f+g)(z) = \sum_{n=0}^{\infty} (a_n + b_n)(z-\alpha)^n, \quad \text{for } |z-\alpha| < r.$$

Multiple Rule If $\lambda \in \mathbb{C}$, then

$$(\lambda f)(z) = \sum_{n=0}^{\infty} \lambda a_n (z-\alpha)^n, \quad \text{for } |z-\alpha| < R.$$

Hence

$$h(z) = 1 + z + \left(2 - \frac{1}{2!}\right) z^2 + \left(2 - \frac{1}{3!}\right) z^3 + \cdots, \quad \text{for } |z|<1.$$

4. Problem 4.2

Find the Taylor series for each of the following functions, about 0.

(a) $h(z) = \mathrm{Log}(1+z) + 3(1-z)^{-1}$

(b) $h(z) = \sin z + \cos z$

5. Taylor series about 0 for $h(z)=(1+z)^{-2}e^z$

Frame 1

$$(1+z)^{-2}=1-2z+3z^2-4z^3+\cdots, \quad \text{for } |z|<1;$$

$$e^z=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots, \quad \text{for } z\in\mathbb{C}.$$

Product Rule Let

$$f(z)=\sum_{n=0}^{\infty}a_n(z-\alpha)^n, \quad \text{for } |z-\alpha|<R,$$

$$g(z)=\sum_{n=0}^{\infty}b_n(z-\alpha)^n, \quad \text{for } |z-\alpha|<R'.$$

If $r=\min\{R,R'\}$, then

$$(fg)(z)=\sum_{n=0}^{\infty}c_n(z-\alpha)^n, \quad \text{for } |z-\alpha|<r,$$

where, for each m, $c_0, c_1, \ldots, c_m$ are the coefficients of $(z-\alpha)^0, (z-\alpha)^1, \ldots, (z-\alpha)^m$ in

$$\left(\sum_{k=0}^{m}a_k(z-\alpha)^k\right)\left(\sum_{k=0}^{m}b_k(z-\alpha)^k\right).$$

$c_0=a_0b_0$
$c_1=a_0b_1+a_1b_0$
$c_2=a_0b_2+a_1b_1+a_2b_0$

Hence

$$h(z)=(1-2z+3z^2-4z^3+\cdots)\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)$$

$$=\underbrace{1}_{c_0}+\underbrace{(1-2)}_{c_1}z+\underbrace{\left(\tfrac{1}{2!}-2+3\right)}_{c_2}z^2+\underbrace{\left(\tfrac{1}{3!}-\tfrac{2}{2!}+3-4\right)}_{c_3}z^3+\cdots$$

up to z^3

$$=1-z+1\tfrac{1}{2}z^2-1\tfrac{5}{6}z^3+\cdots, \quad \text{for } |z|<1.$$

6. Problem 4.3

up to z^3

Find the Taylor series about 0 for:
(a) $h(z)=e^z\sin z$; (b) $h(z)=(\cos z)\mathrm{Log}(1+z)$.

7. Taylor series about 0 for $h(z)=(1+z^2)^{-1}$

$$(1-w)^{-1}=1+w+w^2+w^3+\cdots, \quad \text{for } |w|<1;$$

substitute $w=-z^2$:

$$h(z)=1-z^2+z^4-z^6+\cdots, \quad \text{for } |z|<1.$$

The substitution $w=\lambda z^k$: $\lambda\neq 0, k\in\{1,2,3,\ldots\}$

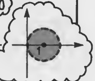
$$|w|<R \iff |z|<\sqrt[k]{R/|\lambda|}$$

8. Taylor series about 1 for $h(z)=\mathrm{Log}\, z$

$$\mathrm{Log}(1+w)=w-\frac{w^2}{2}+\frac{w^3}{3}-\cdots, \quad \text{for } |w|<1;$$ about 0

substitute $w=z-1$:

$$h(z)=(z-1)-\frac{(z-1)^2}{2}+\frac{(z-1)^3}{3}-\cdots, \quad \text{for } |z-1|<1.$$ about 1



The substitution $w=z+(\beta-\alpha)$ changes a Taylor series about β to one about α:

$$|w-\beta|<R \iff |z-\alpha|<R.$$

$w-\beta$
$=z-\alpha$

9. Problem 4.4

Find the Taylor series for:
(a) $h(z)=(1-z^2)^{-1/2}$ about 0;
(b) $h(z)=z^{\alpha}$ about 1, where $\alpha\in\mathbb{C}$.

10. Taylor series about 0 for $h(z)=\text{Log}(\cos z)$

up to z^6

Frame 8

$$\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots, \qquad \text{for } z\in\mathbb{C}; \quad \text{about } 0$$

$$\text{Log}\, w = (w-1)-\frac{(w-1)^2}{2}+\frac{(w-1)^3}{3}-\cdots, \qquad \text{for } |w-1|<1. \quad \text{about } 1=\cos 0$$

Composition Rule Let

$$f(z)=\sum_{n=0}^{\infty} a_n(z-\alpha)^n, \qquad \text{for } |z-\alpha|<R,$$

$$g(w)=\sum_{n=0}^{\infty} b_n(w-\beta)^n, \qquad \text{for } |w-\beta|<R'.$$

If $\beta=f(\alpha)(=a_0)$, then, for some $r>0$,

$$g(f(z))=\sum_{n=0}^{\infty} c_n(z-\alpha)^n, \qquad \text{for } |z-\alpha|<r,$$

where, for each m, $c_0, c_1, \ldots, c_m$ are the coefficients of $(z-\alpha)^0, (z-\alpha)^1, \ldots, (z-\alpha)^m$ in

$$\sum_{k=0}^{m} b_k\left(\sum_{\ell=1}^{m} a_\ell (z-\alpha)^\ell\right)^k$$

$c_0=b_0$
$c_1=b_1a_1$
$c_2=b_1a_2+b_2a_1^2$

Hence, for some $r>0$,

$w-1=(\cos z)-1=-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots$

$$h(z)=\left(-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)-\frac{1}{2}\left(-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)^2+\frac{1}{3}\left(-\frac{z^2}{2!}+\cdots\right)^3+\cdots$$

$$=-\frac{1}{2}z^2+\left(\frac{1}{4!}-\frac{1}{8}\right)z^4+\left(-\frac{1}{6!}+\frac{1}{2!\,4!}-\frac{1}{24}\right)z^6+\cdots$$

$$=-\frac{1}{2}z^2-\frac{1}{12}z^4-\frac{1}{45}z^6+\cdots, \qquad \text{for } |z|<r.$$

11. Problem 4.5

up to z^5

Find the Taylor series for $h(z)=e^{\sin z}$, about 0.

12. Taylor series about 0 for $h(z)=\tan z$

up to z^5

If $f(z)=-\text{Log}(\cos z)$, then $f'(z)=h(z)$ for $|z|<r$;

Frame 10

$$-\text{Log}(\cos z)=\tfrac{1}{2}z^2+\tfrac{1}{12}z^4+\tfrac{1}{45}z^6+\cdots, \qquad \text{for } |z|<r.$$

Theorem 2.2 applied to Taylor series

Differentiation Rule If

$$f(z)=\sum_{n=0}^{\infty} a_n(z-\alpha)^n, \qquad \text{for } |z-\alpha|<R,$$

then f is analytic on $\{z:|z-\alpha|<R\}$, and

$$f'(z)=\sum_{n=1}^{\infty} na_n(z-\alpha)^{n-1}, \qquad \text{for } |z-\alpha|<R.$$

Hence

$$h(z)=z+\tfrac{1}{3}z^3+\tfrac{2}{15}z^5+\cdots, \qquad \text{for } |z|<r.$$

13. Taylor series about 0 for $h(z)=\tan^{-1} z$

up to z^7

If $f(z)=(1+z^2)^{-1}$, then h is a primitive of f on $\{z:|z|<1\}$;

$(1+z^2)^{-1}=1-z^2+z^4-z^6+\cdots, \qquad \text{for } |z|<1.$

Corollary to Theorem 2.2 applied to Taylor series

Integration Rule If

$$f(z)=\sum_{n=0}^{\infty} a_n(z-\alpha)^n, \qquad \text{for } |z-\alpha|<R,$$

then f has primitive F on $\{z:|z-\alpha|<R\}$, where

$$F(z)=\text{constant}+\sum_{n=0}^{\infty}\frac{a_n}{n+1}(z-\alpha)^{n+1}, \qquad \text{for } |z-\alpha|<R.$$

constant $=0$ since $\tan^{-1}0=0$

Hence

$$h(z)=z-\tfrac{1}{3}z^3+\tfrac{1}{5}z^5-\tfrac{1}{7}z^7+\cdots, \qquad \text{for } |z|<1.$$

14. Problem 4.6

Find the Taylor series for $h(z)=\sin^{-1} z$, about 0.

Use Problem 4.4 (a) in Frame 9.

In the audio tape we stated the Product Rule (Frame 5) and the Composition Rule (Frame 10) for Taylor series, but did not prove them. In Problem 4.7, you are asked to complete the proof of the Product Rule. (The proof of the Composition Rule is similar to that for the Product Rule, but it is more complicated: we omit it.)

Problem 4.7

Read the statement of the Product Rule in Frame 5; then complete the boxes in the following proof of the Product Rule.

First notice that we can write

$$f(z) = \sum_{k=0}^{m} a_k(z-\alpha)^k + (z-\alpha)^{m+1}e_f(z), \qquad \text{for } |z-\alpha| < R,$$

$$\text{where } e_f(z) = \sum_{k=0}^{\infty} a_{k+m+1}(z-\alpha)^k.$$

Similarly

$$g(z) = \sum_{k=0}^{m} b_k(z-\alpha)^k + (z-\alpha)^{m+1}e_g(z), \qquad \text{for } |z-r| < R',$$

$$\text{where } e_g(z) = \boxed{}^{(a)}.$$

Thus, if $r = \min\{R, R'\}$, then

$$(fg)(z) = \left(\sum_{k=0}^{m} a_k(z-\alpha)^k\right)\left(\sum_{k=0}^{m} b_k(z-\alpha)^k\right) + (z-\alpha)^{m+1}e(z),$$

$$\text{for } |z-\alpha| < \boxed{}^{(b)},$$

where e is defined by

$$e(z) = e_f(z)\sum_{k=0}^{m} b_k(z-\alpha)^k + e_g(z)\sum_{k=0}^{m} a_k(z-\alpha)^k + (z-\alpha)^{m+1}e_f(z)e_g(z).$$

Since e is $\boxed{}^{(c)}$ on the disc $\{z : |z-\alpha| < r\}$, it has a Taylor series about α, say

$$e(z) = \sum_{k=0}^{\infty} d_k(z-\alpha)^k, \qquad \text{for } |z-\alpha| < r.$$

Thus fg can be represented on $\{z : |z-a| < r\}$ by a power series given by

$$(fg)(z) = \left(\sum_{k=0}^{m} a_k(z-\alpha)^k\right)\left(\sum_{k=0}^{m} b_k(z-\alpha)^k\right)$$

$$+ \boxed{}^{(d)} \qquad (*)$$

Now suppose that the Taylor series about α for fg is

$$(fg)(z) = \sum_{n=0}^{\infty} c_n(z-\alpha)^n, \qquad \text{for } |z-\alpha| < r.$$

Then it follows, from the $\boxed{}^{(e)}$ of the Taylor series representation of a function about a given point, that $c_0, c_1, \ldots, c_m$ are the

coefficients of $(z-\alpha)^0, (z-\alpha)^1, \dots, (z-\alpha)^m$ in

$$\left(\sum_{k=0}^{m} a_k(z-\alpha)^k\right)\left(\sum_{k=0}^{m} b_k(z-\alpha)^k\right),$$

the coefficients of $(z-\alpha)^0, (z-\alpha)^1, \dots, (z-\alpha)^m$ in the final term on the right-hand side of $(*)$ being equal to $\boxed{}^{(f)}$.

4.2 The radius of convergence of a Taylor series

In the audio tape we found the Taylor series for a number of functions. However, we did not specify the radius of convergence of these Taylor series, but simply gave a disc on which we knew that the series converged to the function concerned.

For example, in Frame 12 we used the Differentiation Rule to obtain the Taylor series about 0 for the tangent function:

$$\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots, \qquad \text{for } |z| < r. \tag{4.1}$$

Here, r is some positive number arising from the use of the Composition Rule to obtain the Taylor series about 0 for the function $h(z) = \operatorname{Log}(\cos z)$, in Frame 10.

Now, the tangent function is analytic on its domain $\mathbb{C} - \left\{\left(n+\frac{1}{2}\right)\pi : n \in \mathbb{Z}\right\}$, and so on the open disc $\{z : |z| < \pi/2\}$. Thus, by Taylor's Theorem, it can be represented on $\{z : |z| < \pi/2\}$ by its Taylor series about 0, and so Equation (4.1) can be 'improved' to

$$\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots, \qquad \text{for } |z| < \pi/2. \tag{4.2}$$

Note that this representation of the tangent function is *not* valid on any larger disc with centre 0, since tan is not defined at $\pi/2$.

Representation (4.2) shows that the radius of convergence, R say, of the Taylor series for tan satisfies $R \geq \pi/2$. It seems likely that $R = \pi/2$, but we cannot use the Ratio Test to verify this, since we do not have a formula for the coefficients. Instead we could use the following indirect argument. If $R > \pi/2$, then the function

$$f(z) = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \cdots \qquad (|z| < R)$$

would be analytic and hence continuous on $\{z : |z| < R\}$, and so bounded on the compact set $\{z : |z| \leq \pi/2\}$. This would then imply that tan is bounded on $\{z : |z| < \pi/2\}$, which is false (think of the graph $y = \tan x$ near $\pi/2$). Hence $R = \pi/2$.

See Theorem 5.3 in *Unit A3*.

This argument indicates that it can be tricky to establish the radius of convergence of the Taylor series of a given function, and this is one reason why we do not routinely determine it. Another reason is that, for some functions f, the disc of convergence of the Taylor series for f extends beyond the region on which f is analytic. For example, the function $f(z) = \operatorname{Log} z$ is analytic on the open disc $D = \{z : |z - \alpha| < 1\}$, where $\alpha = -1 + i$, but on no larger open disc with centre α (see Figure 4.3). However, the Differentiation Rule tells us that the Taylor series about α for the functions $f(z) = \operatorname{Log} z$ and $g(z) = 1/z$ must both have the same discs of convergence. This disc must be larger than D since g is analytic on the open disc $E = \{z : |z - \alpha| < \sqrt{2}\}$ (see Figure 4.4).

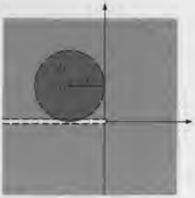


Figure 4.3

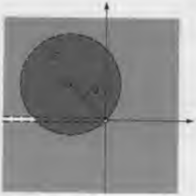


Figure 4.4

We return to examples of this type later in the course when we study *analytic continuation*.

5 THE UNIQUENESS THEOREM

After working through this section, you should be able to:

(a) find the *zeros* of a function and determine their orders;

(b) understand how to use the Uniqueness Theorem.

When defining each of the basic complex functions exp, cos, sin earlier in the course we gave a definition that was designed to generalize the corresponding real function. For example, we defined the complex exponential function by using the formula

$$e^{x+iy} = e^{x}(\cos y + i \sin y).$$

This definition ensures that the complex exponential function agrees with the real exponential function on the real axis. Furthermore, when defined in this way, the complex exponential function has the advantage of being analytic.

Two functions f and g **agree** on a set S if

$$f(z) = g(z), \quad \text{for all } z \in S.$$

One question that this procedure raises is whether we could have defined the exponential function differently. *Is there any other analytic function that agrees with the real exponential function on the real line and which could therefore have been used to define the complex exponential function?*

This raises a more general question that is worthy of investigation.

> If two functions g and h are analytic on a region $\mathcal{R}$ and if g agrees with h on some set $S \subseteq \mathcal{R}$, must g and h agree on $\mathcal{R}$?

S is the set of real numbers $\mathbb{R}$ in the question posed above.

Of course the answer to this question depends on the set S. In this section we show that the answer is yes, provided that S has a limit point in $\mathcal{R}$.

For the definition of limit point, see *Unit A3*, Subsection 3.1.

5.1 Zeros of a function

We begin to answer the question posed in the box above by expressing it in terms of the function $f = g - h$.

> If a function f is analytic on a region $\mathcal{R}$ and if f is zero throughout some set $S \subseteq \mathcal{R}$, must f be zero throughout $\mathcal{R}$?

The word 'throughout' is used here (rather than simply 'on') for emphasis. A function is 'zero throughout a set' if it agrees with the zero function on that set. It is also said to be 'identically zero' on that set.

Clearly it is quite possible for f to be zero at a point $\alpha \in \mathcal{R}$ without having to be zero throughout $\mathcal{R}$. So if S consists of a single point (or, more generally, a finite set of points), then the answer to the question is no. Nevertheless, it is worth pausing to consider this case a little further.

We say that a function f has a *zero* at α if $f(\alpha) = 0$. For such a function the constant term in the Taylor series about α for f is zero:

$$f(z) = f'(\alpha)(z-\alpha) + \frac{f^{(2)}(\alpha)}{2!}(z-\alpha)^2 + \frac{f^{(3)}(\alpha)}{3!}(z-\alpha)^3 + \cdots.$$

When this happens we can factor out $(z - \alpha)$ from the series using the Multiple Rule. Of course, α is still a zero if more of the coefficients are zero. For example, if $f^{(n)}(\alpha) = 0$ for all n up to, but excluding k, then α is a zero of f but we can now factor out $(z - \alpha)^k$ instead of just $(z - \alpha)$. By analogy with polynomial factors, we then say that the zero α is of *order* k.

> **Definition** Let the function f be analytic at α. If
>
> $$f(\alpha) = f'(\alpha) = f''(\alpha) = \ldots = f^{(k-1)}(\alpha) = 0, \quad \text{but} \quad f^{(k)}(\alpha) \neq 0,$$
>
> then f has a **zero at α of (finite) order k.**
>
> A zero of order 1 is often called a **simple zero.**

For example, the function sin has a simple zero at π since

$$\sin\pi = 0 \quad \text{and} \quad \sin'\pi = \cos\pi \neq 0.$$

By contrast, the function $f(z) = 1 + \cos z$ has a zero of order 2 at π since

$$f(\pi) = 0, \quad f'(\pi) = -\sin\pi = 0 \quad \text{and} \quad f''(\pi) = -\cos\pi \neq 0.$$

On the other hand, the function $f(z) = 0$ has a zero at each point α of $\mathbb{C}$, but

$$f(\alpha) = f'(\alpha) = f''(\alpha) = \ldots = 0;$$

so none of the zeros is of finite order.

We usually classify the zeros of a function f without calculating its higher derivatives, by using the following theorem.

> **Theorem 5.1** A function f is analytic at a point α, and has a zero of order k at α if, and only if, for some $r > 0$,
>
> $$f(z) = (z-\alpha)^k g(z), \qquad \text{for } |z - \alpha| < r,$$
>
> where the function g is analytic at α, and $g(\alpha) \neq 0$.

Proof If f is analytic at α, and has a zero of order k at α, then the Taylor series about α for f has the form

$$\begin{aligned} f(z) &= 0+0+\cdots+0+\frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k \\ &\quad + \frac{f^{(k+1)}(\alpha)}{(k+1)!}(z-\alpha)^{k+1}+\cdots, \qquad \text{for } |z-\alpha|<r, \\ &= (z-\alpha)^k g(z), \end{aligned}$$

where r is some positive real number, and

$$g(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^{n-k}, \qquad \text{for } |z-\alpha|<r.$$

By the Differentiation Rule for power series, g is analytic at α. Furthermore,

$$g(\alpha) = f^{(k)}(\alpha)/k! \neq 0.$$

Conversely, if g is analytic and non-zero at α, then

$$g(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n, \qquad \text{for } |z-\alpha|<r,$$

where r is some positive real number, and $a_0 \neq 0$. But then

$$f(z) = (z-\alpha)^k g(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^{n+k}, \qquad \text{for } |z-\alpha|<r,$$

so that, by the uniqueness of Taylor series,

$$f(\alpha) = f'(\alpha) = \ldots = f^{(k-1)}(\alpha) = 0,$$

and

$$f^{(k)}(\alpha) = k!a_0 \neq 0.$$

Hence f has a zero at α of order k. ■

This theorem tells us that we can classify the zeros of a function at α by factoring out the appropriate power of $(z-\alpha)$, to leave an analytic function that is non-zero at α, as we now illustrate.

Example 5.1

Classify the zeros of the function

$$f(z) = (z-2)^3(z^2+1)(z-i)e^z.$$

Solution

Since $f(z)=0$ when $z=2$, i and $-i$, it follows that f has zeros at 2, i, and $-i$.

Now

$$f(z) = (z-2)^3 g(z),$$

where $g(z) = (z^2+1)(z-i)e^z$ is a function that is analytic but non-zero at 2. Hence f has a zero of order 3 at 2. Similarly, f has a zero of order 2 at i, and a simple zero at $-i$. ■

Problem 5.1

Classify the zeros of each of the following functions f.

(a) $f(z) = z^3(z-1)^4(z+2)$

(b) $f(z) = (z-3)/(z+2)$

(c) $f(z) = (z^2+9)^3e^{-z}/(z^2+4)$

If after factoring out all the obvious $(z-\alpha)$ terms, you are still left with a function that is zero at α, then you can deal with the remaining function by finding its Taylor series about α.

Example 5.2

Classify the zero of the function $f(z) = (z-1)\,\mathrm{Log}\,z$ at 1.

Solution

Although we can factor out $(z-1)$, that still leaves the function $\mathrm{Log}\,z$ which is zero at 1. So, using the Taylor series about 1 for Log, we obtain

$$\begin{aligned} f(z) &= (z-1)\,\mathrm{Log}\,z \\ &= (z-1)\left((z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots\right), \qquad \text{for } |z-1|<1, \\ &= (z-1)^2\left(1 - \frac{z-1}{2} + \frac{(z-1)^2}{3} - \cdots\right) \\ &= (z-1)^2 \times \text{ (a function } g \text{ that is analytic but non-zero at 1).} \end{aligned}$$

Here
$$g(z) = 1 - \frac{z-1}{2} + \frac{(z-1)^2}{3} - \cdots.$$

Thus f has a zero of order 2 at 1. ■

Problem 5.2

For each of the following functions f, find the order of the zero of f at 0.

(a) $f(z) = z^4 \sin 2z$

(b) $f(z) = z^2(\cos z - 1)$

(c) $f(z) = 6\sin(z^2) + z^2(z^4 - 6)$

Earlier we pointed out that a function that is analytic on a region $\mathcal{R}$ can have a zero in $\mathcal{R}$ without having to be zero throughout $\mathcal{R}$. The following theorem shows that this is the case only if the zero is of finite order.

Theorem 5.2 If the function f is analytic on a region $\mathcal{R}$ and not identically zero there, then any zero of f is of finite order.

Proof If α is a zero of f which is not of finite order, then all the higher derivatives of f must be zero at α. Thus we need to prove that

This proof may be omitted on a first reading.

if, for some $\alpha \in \mathcal{R}$, $f^{(n)}(\alpha) = 0$, for $n = 0, 1, 2, \ldots$, then
$f(z) = 0$, for all $z \in \mathcal{R}$.

First consider the special case when $\mathcal{R}$ is an open disc, say $D = \{z : |z-\beta| < r\}$. If $\alpha \in D$, then $r' = r - |\beta - \alpha| > 0$ (see Figure 5.1) and we can construct a finite sequence of open discs

$$D_k = \{z : |z - \alpha_k| < r'\}, \qquad k = 0, 1, 2, \ldots, m,$$

where

$$\alpha_k = \alpha + \frac{k}{m}(\beta - \alpha), \qquad k = 0, 1, 2, \ldots, m,$$

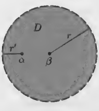


Figure 5.1

and m is so large that $|\beta - \alpha|/m < r'$. Then $\alpha_0 = \alpha$, $\alpha_m = \beta$ and, for $k = 1, 2, \ldots, m$, the centre of the disc D_k lies in D_{k-1} (see Figure 5.2).

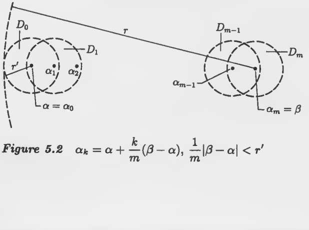


Figure 5.2 $\alpha_k = \alpha + \dfrac{k}{m}(\beta - \alpha)$, $\dfrac{1}{m}|\beta - \alpha| < r'$

Since the Taylor series about $\alpha = \alpha_0$ for f has zero coefficients, we deduce that

$$f(z) = 0, \qquad \text{for } z \in D_0,$$

and hence that

$$f^{(n)}(\alpha_1) = 0, \qquad \text{for } n = 0, 1, 2, \ldots,$$

since $\alpha_1 \in D_0$. Repeating this argument in $D_1, D_2, \ldots, D_{m-1}$, we deduce that

$$f^{(n)}(\beta) = 0, \qquad \text{for } n = 0, 1, 2, \ldots .$$

Thus, using the Taylor series about β for f, we find that

$$f(z) = 0, \qquad \text{for } z \in D,$$

as required.

Now let $\mathcal{R}$ be a general region and suppose that $\beta \in \mathcal{R}$ with $\beta \neq \alpha$. Then we can join α to β by a path $\Gamma : \gamma(t)$ $(t \in [a, b])$ lying in $\mathcal{R}$ (so that $\gamma(a) = \alpha$, $\gamma(b) = \beta$) and then apply the Paving Theorem. This provides a finite sequence of discs $D_k, k = 1, 2, \ldots, n$, in $\mathcal{R}$ and a finite sequence of points $t_0, t_1, \ldots, t_n$, with

The Paving Theorem was discussed in *Unit B1*, Section 3.

$$a = t_0 < t_1 < \ldots < t_n = b,$$

such that

$$\gamma([t_{k-1}, t_k]) \subseteq D_k, \qquad \text{for } k = 1, 2, \ldots, n.$$

(See Figure 5.3.)

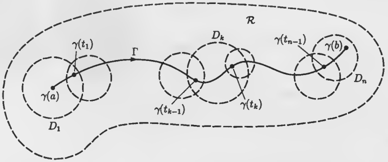


Figure 5.3

Since $f^{(n)}(\alpha) = 0$, for $n = 0, 1, 2, \ldots$, and $\alpha = \gamma(t_0) \in D_1$, it follows, by the special case above, that $f(z) = 0$, for $z \in D_1$, and hence that

$$f^{(n)}(\gamma(t_1)) = 0, \qquad \text{for } n = 0, 1, 2, \ldots .$$

Repeating this argument with $D_2, D_3, \ldots, D_n$, we deduce that $f(\gamma(t_n)) = f(\beta) = 0$. Since $\beta \in \mathcal{R}$ was arbitrary, the proof is complete. ■

We are now well on the way to answering the question posed at the beginning of this subsection. For if f is analytic on a region $\mathcal{R}$ and has a zero in $\mathcal{R}$ that is not of finite order, then f must be identically zero on $\mathcal{R}$.

But how do we recognize a zero that is not of finite order? To help us do this we make the following definition.

Definition A zero α of a function f is said to be **isolated** if some disc centred at α contains no other zeros of f.

We then make the following observation.

> **Theorem 5.3 Isolated Zeros**
>
> A zero of finite order is isolated.

Proof Suppose that α is a zero of f of order k. Then, by Theorem 5.1,

$$f(z) = (z - \alpha)^k g(z),$$

where g is analytic at α, and $g(\alpha) \neq 0$. Since g is analytic at α, it must be continuous at α, and so $g(z)$ is non-zero on some open disc with centre α. Thus $f(z)$ is non-zero on the same open disc with centre α, except at α itself. Thus α is an isolated zero. ■

Now, a zero cannot be isolated if it is the limit point of a set of zeros. We can therefore answer the question posed at the beginning of this subsection, as follows.

> **Theorem 5.4** If the function f is analytic on a region $\mathcal{R}$ and if S is a set of zeros of f with a limit point in $\mathcal{R}$, then f is identically zero on $\mathcal{R}$.

The limit point does not have to be in S for the theorem to apply but it does have to be in $\mathcal{R}$.

Proof Let $\{z_n\}$ be a sequence of zeros in S (see Figure 5.4) which converges to a limit point α of S in $\mathcal{R}$. Since f is continuous at α, it follows that $f(\alpha) = \lim_{n\to\infty} f(z_n) = 0$, and so α is a zero of f in $\mathcal{R}$. Furthermore, the zero is not isolated because there is some zero z_n in *any* disc centred at α. By Theorem 5.3, α is not of finite order, and so, by Theorem 5.2, f is identically zero on $\mathcal{R}$. ■

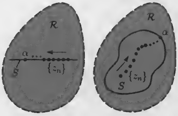


Figure 5.4

An immediate consequence of this theorem is the Uniqueness Theorem.

> **Theorem 5.5 Uniqueness Theorem**
>
> Let the functions f and g be analytic on a region $\mathcal{R}$ and suppose that f agrees with g throughout a set $S \subseteq \mathcal{R}$, where S has a limit point in $\mathcal{R}$. Then f agrees with g throughout $\mathcal{R}$.

We can now return to the question that we used to motivate our discussion of the Uniqueness Theorem in the introduction to this section.

Example 5.3

Show that

$$f(x + iy) = e^x(\cos y + i \sin y),$$

defines a unique entire function that agrees with the real exponential function on $\mathbb{R}$.

Solution

Suppose that g is any other entire function that agrees with the real exponential function on $\mathbb{R}$. We apply the Uniqueness Theorem with $S = \mathbb{R}$ and $\mathcal{R} = \mathbb{C}$. Since $\mathbb{R}$ has a limit point in $\mathbb{C}$ (for example, 0) and

$$f(x) = e^x = g(x), \qquad \text{for } x \in \mathbb{R},$$

it follows from the Uniqueness Theorem that

$$f(z) = e^z = g(z), \qquad \text{for } z \in \mathbb{C}. \quad ■$$

Problem 5.3

Show that if the functions f and g are analytic on a region $\mathcal{R}$ and are represented by the same Taylor series on an open disc $D \subseteq \mathcal{R}$, then f agrees with g on $\mathcal{R}$.

To apply Theorem 5.5, S does not have to be the real line or an open disc. Any set with a limit point in $\mathcal{R}$ will do.

Problem 5.4

Let f and g be entire functions. Which of the following conditions are sufficient to ensure that f and g are equal?

(a) f and g agree on the set $S = \{z : |z| = 2\}$

(b) f and g agree on the set of positive integers $S = \mathbb{N}$

(c) f and g agree on the set $S = \{1/n : n \in \mathbb{N}\}$

Problem 5.5

Prove that if f is an entire function and

$$f(1/n) = 1/n, \qquad \text{for } n = 1, 2, \ldots,$$

then

$$f(z) = z, \qquad \text{for } z \in \mathbb{C}.$$

5.2 Using power series to define functions

We began this section with a reminder about the way we used real functions to define the basic complex functions exp, sin, cos. However there is an alternative way to define these basic functions that avoids the need to use real functions at all, and this is to give a definition in terms of power series. We have devoted this final subsection to a brief summary of this alternative approach, which is adopted in many complex analysis texts.

Starting with the exponential function, we *define*

$$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \qquad (z \in \mathbb{C}).$$

Since this power series has infinite radius of convergence, we know that this function is entire.

Having defined the exponential function in this way, we can check that it has all the properties we wish it to have. For example, using the Differentiation Rule we can differentiate the power series term by term and check that the exponential function is its own derivative:

$$\exp' = \exp.$$

Next we can find the Taylor series for exp about any point α. We have:

$$\begin{aligned}
\exp z &= \exp\alpha + (\exp'\alpha)(z-\alpha) + (\exp''\alpha)\frac{(z-\alpha)^2}{2!} + (\exp'''\alpha)\frac{(z-\alpha)^3}{3!} + \cdots \\
&= \exp\alpha\left(1 + (z-\alpha) + \frac{(z-\alpha)^2}{2!} + \frac{(z-\alpha)^3}{3!} + \cdots\right) \\
&= \exp\alpha \exp(z-\alpha).
\end{aligned}$$

On substituting $z = \alpha + \beta$, we obtain the exponential addition rule:

$$\exp(\alpha + \beta) = \exp\alpha \exp\beta, \qquad \text{for } \alpha, \beta \in \mathbb{C}.$$

The trigonometric and hyperbolic functions can then be defined in terms of the exponential function as we did earlier in the course:

$$\sin z = \frac{\exp(iz) - \exp(-iz)}{2i}, \quad \cos z = \frac{\exp(iz) + \exp(-iz)}{2}, \quad \tan z = \frac{\sin z}{\cos z};$$

and

$$\sinh z = \frac{\exp z - \exp(-z)}{2}, \quad \cosh z = \frac{\exp z + \exp(-z)}{2}, \quad \tanh z = \frac{\sinh z}{\cosh z}.$$

The trigonometric and hyperbolic identities then follow as in *Unit A2.* Also the derivatives of the trigonometric and hyperbolic functions are found as in *Unit A4.* Since the same derivatives are obtained as in our original treatment, the Taylor series must be the same as well.

A slight difficulty arises when we try to define the function Log to be the inverse of the exponential function. Before we can do this we need to show that exp is one-one on the strip $\{z : -\pi < \operatorname{Im} z \leq \pi\}$. The difficulty lies in the fact that without the theory of real trigonometry we do not know what the number π means, so we have to define π before we can proceed further.

Although we do not wish to discuss the technicalities of this definition, the idea is to define π to be the smallest positive real solution of the equation $\sin z = 0$. We can then use the Taylor series for sin and the trigonometric identities to show that sin and cos have all the familiar properties when restricted to the real line, including being real and having period 2π.

We can then use the identity

$$\exp(x + iy) = \exp x \exp iy = \exp x(\cos y + i \sin y)$$

to show that exp is one-one on the strip $\{z : -\pi < \operatorname{Im} z \leq \pi\}$. This enables us to define Log to be the inverse of exp on this strip, as in *Unit A2.*

Finally we define the principal αth power of z by

$$z^\alpha = \exp(\alpha \operatorname{Log} z) \qquad (z \neq 0).$$

If we define e to be the number $\exp 1$, then $\operatorname{Log} e = 1$. This shows that

$$e^\alpha = \exp \alpha, \text{ as expected.}$$

So in this subsection we have indicated how the basic functions of complex analysis can be defined starting with the power series definition of the exponential function.

Of course, we need not restrict ourselves to the exponential function because Taylor's Theorem tells us that *any* analytic function can be represented by a power series (on some disc).

Many analytic functions are not readily expressible in terms of the basic complex functions, and it is often easier to specify such functions in terms of power series. For example, the analytic solutions of some differential equations are best found by substituting into the equation an arbitrary power series (using the Differentiation Rule for the derivatives). We can then equate the coefficients of corresponding powers of z and hence obtain a recurrence relation for the coefficients of the power series. This series gives a solution of the equation that is valid on the disc of convergence of the series.

Although a power series determines a function only on the disc of convergence of the series, the Uniqueness Theorem ensures that this is sufficient to determine an analytic function throughout its domain. Later in the course we show how we can extend the specification of a function on a disc to the whole of its domain by using a technique known as *analytic continuation.*

EXERCISES

Section 1

Exercise 1.1 For each of the following series, calculate the 0th, 1st, 2nd, 3rd and nth partial sums.

(a) $\sum_{n=0}^{\infty} i$ (b) $\sum_{n=0}^{\infty} \frac{i}{10^n}$ (c) $\sum_{n=0}^{\infty} i^n$

Exercise 1.2 For each of the following series, determine whether it is convergent or divergent. If the series converges, give its sum.

(a) $\sum_{n=0}^{\infty} \frac{1}{\sqrt{2}}(1-i)^n$ (b) $\sum_{n=0}^{\infty} \left(\frac{1+i}{2}\right)^n$ (c) $\sum_{n=0}^{\infty} \left(\frac{1-i}{\sqrt{2}}\right)^n$

(d) $\sum_{n=2}^{\infty} \binom{n}{2} i^n$

Exercise 1.3 Use Theorem 1.5 and the result of Exercise 1.2(b) to show that

$$\sum_{n=0}^{\infty} 2^{-n/2} \cos \frac{n\pi}{4} = 1.$$

Exercise 1.4 Determine which of the following series are absolutely convergent.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^2+i}$ (c) $\sum_{n=1}^{\infty} \frac{2^n - i}{n^2}$

(d) $\sum_{n=1}^{\infty} \frac{i^n}{n\sqrt{n}}$ (e) $\sum_{n=0}^{\infty} e^{n(i-1)}$

Section 2

Exercise 2.1 For each of the following power series, determine the radius of convergence and the disc of convergence.

(a) $\sum_{n=0}^{\infty} (-z)^n$ (b) $\sum_{n=0}^{\infty} (3iz)^n$ (c) $\sum_{n=0}^{\infty} (3i-z)^n$

(d) $\sum_{n=0}^{\infty} (2z-i)^n$ (e) $\sum_{n=0}^{\infty} nz^n$ (f) $\sum_{n=1}^{\infty} n!(z+1)^n$

(g) $\sum_{n=1}^{\infty} \left(\frac{z}{n}\right)^n$ (h) $\sum_{n=1}^{\infty} \frac{(z-\pi)^n}{n!}$

Exercise 2.2 For each of the following power series, find the disc of convergence and the sum function on that disc.

(a) $\sum_{n=0}^{\infty} (2z)^n$ (b) $\sum_{n=0}^{\infty} (n+1)z^n$ (c) $\sum_{n=0}^{\infty} (n+1)(n+2)z^n$

(d) $\sum_{n=1}^{\infty} \frac{z^n}{n}$

Section 3

Exercise 3.1 Use Taylor's Theorem to find the Taylor series for each of the following functions, about the given point. In each case give the general term of the series.

(a) $f(z) = \sinh 2z$, about 0

(b) $f(z) = z \sin z$, about 0

(c) $f(z) = e^{iz}$, about $\pi/4$

Section 4

Exercise 4.1 Use the methods of Section 4 to find the Taylor series (to the given power of z) about 0 for each of the following functions, and state an open disc on which each series converges to the given function.

(a) $f(z) = (1+z)^{1/2}$ (up to z^3)

(b) $f(z) = (1+z)^{1/2} - (1-z)^{1/2}$ (up to z^3)

(c) $f(z) = \operatorname{Log}\left(\dfrac{1-z}{1+z}\right)$ (up to z^3)

(d) $f(z) = z^3 \cos(z^2)$ (up to z^{11})

(e) $f(z) = (\sin z)(\cos z)$ (up to z^5)

(f) $f(z) = \operatorname{Log}(\cosh z)$ (up to z^6)

(g) $f(z) = \tanh z$ (up to z^5) (*Hint*: Use part (f).)

Exercise 4.2 Repeat Exercise 3.1 (a) and (b) but use the methods of Section 4 rather than Taylor's Theorem.

Section 5

Exercise 5.1 Use Theorem 5.1 to classify the zeros of each of the following functions.

(a) $f(z) = z^2(z^2+4)^3$ (b) $f(z) = z \sin z$

(*Hint*: In part (b), the following formula will be useful:

$$\begin{aligned}\sin(z - k\pi) &= \sin z \cos(k\pi) - \cos z \sin(k\pi)\\ &= (-1)^k \sin z, \quad k \in \mathbb{Z}.)\end{aligned}$$

SOLUTIONS TO THE PROBLEMS

Section 1

1.1 We have

$$s_0 = 1$$
$$s_1 = 1 + \frac{i}{2}$$
$$s_2 = 1 + \frac{i}{2} + \left(\frac{i}{2}\right)^2 = \frac{3}{4} + \frac{i}{2}$$
$$s_3 = 1 + \frac{i}{2} + \left(\frac{i}{2}\right)^2 + \left(\frac{i}{2}\right)^3 = \frac{3}{4} + \frac{3i}{8}$$

1.2 (a) Here

$$\begin{aligned} s_n &= \frac{i}{4} + \left(\frac{i}{4}\right)^2 + \cdots + \left(\frac{i}{4}\right)^n \\ &= \frac{i}{4}\left(1 + \frac{i}{4} + \cdots + \left(\frac{i}{4}\right)^{n-1}\right) \\ &= \frac{i}{4}\,\frac{1-\left(\frac{i}{4}\right)^n}{1-\frac{i}{4}} \\ &= \frac{i}{4-i}\left(1-\left(\frac{i}{4}\right)^n\right). \end{aligned}$$

Since $\{(i/4)^n\}$ is a basic null sequence, we have

$$\lim_{n\to\infty} s_n = \frac{i}{4-i} = \frac{i(4+i)}{17} = \frac{-1+4i}{17},$$

and so the series is convergent with sum $(-1+4i)/17$.

(b) Here

$$\begin{aligned} s_n &= -7i(1 + (-i) + (-i)^2 + \cdots + (-i)^{n-1}) \\ &= -7i\frac{1-(-i)^n}{1-(-i)}. \end{aligned}$$

Since the sequence $\{(-1)^n\}$ is divergent (by Theorem 1.7(b), *Unit A3*), $\lim\limits_{n\to\infty} s_n$ does not exist, and so the series is divergent.

(c) Here

$$\begin{aligned} s_n &= \left(\frac{1-i}{2}\right) + \left(\frac{1-i}{2}\right)^2 + \cdots + \left(\frac{1-i}{2}\right)^n \\ &= \left(\frac{1-i}{2}\right)\left(1 + \left(\frac{1-i}{2}\right) + \cdots + \left(\frac{1-i}{2}\right)^{n-1}\right) \\ &= \left(\frac{1-i}{2}\right)\frac{1-\left(\frac{1-i}{2}\right)^n}{1-\left(\frac{1-i}{2}\right)} \\ &= \left(\frac{1-i}{1+i}\right)\left(1-\left(\frac{1-i}{2}\right)^n\right). \end{aligned}$$

Since $|(1-i)/2| < 1$, the sequence $\{((1-i)/2)^n\}$ converges to 0. Hence

$$\lim_{n\to\infty} s_n = \frac{1-i}{1+i} = \frac{(1-i)(1-i)}{2} = -i,$$

and so the series is convergent with sum $-i$.

1.3 (a) Since $|1+i| > 1$, the terms of the series $\sum\limits_{n=1}^{\infty}(1+i)^n$ tend to infinity. By the Non-null Test, the series diverges.

(b) By the First Subsequence Rule, the terms of the series $\sum\limits_{n=1}^{\infty} i(-1)^n$ form a divergent sequence; for the even terms tend to i, whereas the odd terms tend to $-i$. By the Non-null Test, the series diverges.

(c) Since $1/n! \le 1/n$, it follows from the Squeeze Rule for sequences that $\{1/n!\}$ is a null sequence. By the Reciprocal Rule for sequences, the terms of the series $\sum\limits_{n=1}^{\infty} n!$ tend to infinity, and so the series diverges by the Non-null Test.

(d) By the Combination Rules for sequences,

$$\frac{n^2+i}{2n^2+n+3} = \frac{1+i/n^2}{2+1/n+3/n^2} \to \frac{1}{2} \text{ as } n\to\infty.$$

So, by the Non-null Test, the series

$$\sum_{n=1}^{\infty} \frac{n^2+i}{2n^2+n+3}$$

diverges.

1.4 (a) Here $a = z = i/4$. Since $|i/4| = 1/4 < 1$, the series is convergent. The sum is

$$\frac{i/4}{1-i/4} = \frac{i}{4-i} = \frac{i(4+i)}{17} = \frac{-1+4i}{17}.$$

(b) Divergent, since $|-i| = 1$.

(c) Convergent, since $|(1-i)/2| < 1$. The sum is

$$\frac{(1-i)/2}{1-(1-i)/2} = \frac{1-i}{1+i} = \frac{(1-i)(1-i)}{2} = -i.$$

1.5 We have

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda z_k &= \lim_{n\to\infty} \sum_{k=1}^{n} \lambda z_k \\ &= \lim_{n\to\infty} \lambda \sum_{k=1}^{n} z_k \\ &= \lambda \lim_{n\to\infty} \sum_{k=1}^{n} z_k \qquad \text{(Multiple Rule for sequences)} \\ &= \lambda s. \end{aligned}$$

1.6 Here

$$\cos nx = \operatorname{Re}\left(e^{inx}\right), \quad \text{for } n = 0, 1, 2, \ldots .$$

So, by Theorems 1.2 and 1.5, and the calculation in Example 1.2,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{2^n}\cos nx &= \operatorname{Re}\left(\sum_{n=0}^{\infty} \frac{1}{2^n} e^{inx}\right) \\ &= \operatorname{Re}\left(\frac{(1-\frac{1}{2}\cos x) + \frac{1}{2}i\sin x}{\frac{5}{4}-\cos x}\right) \\ &= \frac{4-2\cos x}{5-4\cos x}. \end{aligned}$$

1.7 Here

$$\left|\frac{\cos n}{n\sqrt{n}}\right| \le \frac{1}{n^{3/2}}, \quad \text{for } n = 1, 2, \ldots .$$

Furthermore $\sum\limits_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent, by Theorem 1.3, so

$$\sum_{n=1}^{\infty} \frac{\cos n}{n\sqrt{n}}$$

is convergent, by the Comparison Test.

1.8 Let $a_n = |z_n|$. Then $\sum_{n=1}^{\infty} a_n$ is a convergent series. Furthermore

$$|z_n| \le a_n, \quad \text{for } n = 1, 2, \ldots .$$

So, by the Comparison Test, $\sum_{n=1}^{\infty} z_n$ is convergent.

1.9 (a) In this case

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges, by Theorem 1.3. It follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is *not* absolutely convergent.

(b) Here

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n (1+i)^n}{2^n} \right| = \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}} \right)^n$$

is a convergent geometric series. Thus

$$\sum_{n=0}^{\infty} \frac{(-1)^n (1+i)^n}{2^n}$$

is absolutely convergent, by the Absolute Convergence Test.

1.10 Let $\{s_n\}$ be the sequence of the partial sums for the series. Then

$$s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + e_n,$$

where

$$e_n = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1/n, & \text{if } n \text{ is odd.} \end{cases}$$

Now $|e_n| \le 1/n$, for $n = 1, 2, \ldots$, so $\{e_n\}$ is a null sequence. Furthermore

$$0 \le \left(\frac{1}{n} - \frac{1}{(n+1)} \right) = \frac{1}{n(n+1)} \le \frac{1}{n^2},$$

so, in the expression for s_n, the sum of the terms in the brackets increases with n and is bounded above by $\sum_{n=1}^{\infty} \frac{1}{n^2}$. By the Monotone Convergence Theorem, the sum of the terms in brackets converges to l, say.

Since $\{e_n\}$ is null, $\{s_n\}$ must also converge to l, and so

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is convergent.

1.11 (a) Let $z_n = \dfrac{n^2}{3^n + i}$. Then

$$\begin{aligned} \left| \frac{z_{n+1}}{z_n} \right| &= \left| \frac{(n+1)^2}{3^{n+1} + i} \right| \div \left| \frac{n^2}{3^n + i} \right| \\ &= \left| \frac{3^n + i}{3^{n+1} + i} \right| \left(\frac{n+1}{n} \right)^2 \\ &= \left| \frac{1 + i/3^n}{3 + i/3^n} \right| \left(1 + \frac{1}{n} \right)^2 . \end{aligned}$$

Using the Combination Rules for sequences and the continuity of the modulus function, we see that $|z_{n+1}/z_n|$ tends to $1/3$ as $n \to \infty$. It follows from the Ratio Test that $\sum_{n=1}^{\infty} \frac{n^2}{3^n + i}$ is absolutely convergent.

(b) Let $z_n = \dfrac{z^n}{n!}$. Then

$$\left| \frac{z_{n+1}}{z_n} \right| = \left| \frac{z^{n+1}}{(n+1)!} \right| \div \left| \frac{z^n}{n!} \right| = \frac{|z|}{n+1},$$

which tends to 0 as $n \to \infty$. It follows from the Ratio Test that $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is absolutely convergent (and hence convergent) for all $z \in \mathbb{C}$.

Section 2

2.1 (a) This geometric series converges with sum $(1 - 4z)^{-1}$ when $|4z| < 1$, and diverges when $|4z| > 1$. Since

$$|4z| < 1 \iff |z| < \tfrac{1}{4} \quad \text{and} \quad |4z| > 1 \iff |z| > \tfrac{1}{4},$$

the radius of convergence is $\frac{1}{4}$.

(b) This geometric series converges with sum $1/(1 - \alpha z)$ when $|\alpha z| < 1$, and diverges when $|\alpha z| > 1$. Since

$$|\alpha z| < 1 \iff |z| < 1/|\alpha| \quad \text{and} \quad |\alpha z| > 1 \iff |z| > 1/|\alpha|,$$

the radius of convergence is $1/|\alpha|$.

2.2 (a) The ratio of the $(n+1)$th and nth terms (with $z \ne 0$) is

$$\frac{|(2^{n+1} + 4^{n+1})z^{n+1}|}{|(2^n + 4^n)z^n|} = \frac{2(1/2)^n + 4}{(1/2)^n + 1} |z|,$$

which tends to $4|z|$ as $n \to \infty$. So, by the Ratio Test, the power series converges for $4|z| < 1$ and diverges for $4|z| > 1$. The radius of convergence of the series is therefore $1/4$.

(b) The ratio of the $(n+1)$th and nth terms (with $z \ne -7$) is

$$\begin{aligned} &\left| \frac{(2n+2)!(z+7)^{n+1}}{(n+1)!} \right| \div \left| \frac{(2n)!(z+7)^n}{n!} \right| \\ &= \frac{(2n+2)(2n+1)}{n+1} |z+7| = 2(2n+1)|z+7|, \end{aligned}$$

which tends to ∞ as $n \to \infty$. So, by the Ratio Test, the power series diverges for all $z \in \mathbb{C} - \{-7\}$. The radius of convergence of the series is therefore 0.

(c) The ratio of the $(n+1)$th and nth terms (with $z \ne 1$) is

$$\begin{aligned} &\frac{|((n+1) + 2^{-n-1})(z-1)^{n+1}|}{|(n + 2^{-n})(z-1)^n|} \\ &= \frac{1 + 1/n + (1/n)2^{-n-1}}{1 + (1/n)2^{-n}} |z - 1|, \end{aligned}$$

which tends to $|z - 1|$ as $n \to \infty$. So, by the Ratio Test, the power series converges for $|z - 1| < 1$ and diverges for $|z - 1| > 1$. The radius of convergence of the series is therefore 1.

2.3 The discs of convergence for the series in Example 2.1 are:

(a) the empty set $\varnothing$;

(b) the complex plane $\mathbb{C}$;

(c) the open disc $\{z : |z| < 1\}$.

The discs of convergence for the series in Problem 2.2 are:

(a) the open disc $\{z : |z| < 1/4\}$;

(b) the empty set $\varnothing$;

(c) the open disc $\{z : |z-1| < 1\}$.

2.4 (a) Here the ratio of the $(n+1)$th and nth terms is

$$\left|\frac{z^{n+1}}{(n+1)^2}\right| \div \left|\frac{z^n}{n^2}\right| = \frac{n^2}{(n+1)^2}|z| = \frac{1}{(1+1/n)^2}|z|,$$

which tends to $|z|$ as $n \to \infty$. So, by the Ratio Test, the radius of convergence is 1.

(b) Here the ratio of the $(n+1)$th and nth terms is

$$\left|\frac{z^{n+1}}{n+1}\right| \div \left|\frac{z^n}{n}\right| = \frac{n}{n+1}|z| = \frac{1}{1+1/n}|z|,$$

which tends to $|z|$ as $n \to \infty$. So, by the Ratio Test, the radius of convergence is 1.

Section 3

3.1 Each of the functions f in this problem is entire, so its Taylor series about 0 must converge to $f(z)$ for all $z \in \mathbb{C}$.

(a) We have

$$\begin{aligned} f(z) &= \sin z, & &\text{so} & f(0) &= 0; \\ f'(z) &= \cos z, & &\text{so} & f'(0) &= 1; \\ f^{(2)}(z) &= -\sin z, & &\text{so} & f^{(2)}(0) &= 0; \\ f^{(3)}(z) &= -\cos z, & &\text{so} & f^{(3)}(0) &= -1; \\ f^{(4)}(z) &= \sin z, & &\text{so} & f^{(4)}(0) &= 0. \end{aligned}$$

Since every fourth derivative brings us back to the sin function, the above pattern repeats itself. The Taylor series about 0 for the function $f(z) = \sin z$ is:

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

(b) We have

$$\begin{aligned} f(z) &= \cosh z, & &\text{so} & f(0) &= 1; \\ f'(z) &= \sinh z, & &\text{so} & f'(0) &= 0; \\ f^{(2)}(z) &= \cosh z, & &\text{so} & f^{(2)}(0) &= 1. \end{aligned}$$

Since every second derivative brings us back to the cosh function, the above pattern repeats itself. The Taylor series about 0 for the function $f(z) = \cosh z$ is

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

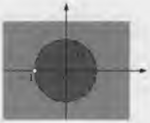
(c) We have

$$\begin{aligned} f(z) &= \sinh z, & &\text{so} & f(0) &= 0; \\ f'(z) &= \cosh z, & &\text{so} & f'(0) &= 1; \\ f^{(2)}(z) &= \sinh z, & &\text{so} & f^{(2)}(0) &= 0. \end{aligned}$$

Since every second derivative brings us back to the sinh function, the above pattern repeats itself. The Taylor series about 0 for the function $f(z) = \sinh z$ is

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

3.2 The function $f(z) = (1+z)^{-3}$ is analytic on the region $\mathbb{C} - \{-1\}$. The largest open disc, centred at 0, that will fit in this region is $D = \{z : |z| < 1\}$. So, by Taylor's Theorem, the Taylor series about 0 for f converges to $f(z)$ for $|z| < 1$.



The Taylor series is found by calculating the higher derivatives of f at 0. We have

$$\begin{aligned} f(z) &= (1+z)^{-3}, \\ f'(z) &= -3(1+z)^{-4}, \\ f^{(2)}(z) &= (-4)(-3)(1+z)^{-5}, \\ f^{(3)}(z) &= (-5)(-4)(-3)(1+z)^{-6}. \end{aligned}$$

It follows that

$$f(0) = 1,\ f'(0) = -3,\ f^{(2)}(0) = (4)(3),$$
$$f^{(3)}(0) = -(5)(4)(3), \ldots .$$

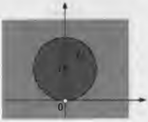
In general,

$$\frac{f^{(n)}(0)}{n!} = (-1)^n \frac{(n+2)(n+1)}{2}, \quad \text{for } n = 0, 1, 2, \ldots .$$

The Taylor series about 0 for f is therefore

$$(1+z)^{-3} = 1 - \frac{(3)(2)}{2}z + \frac{(4)(3)}{2}z^2 - \frac{(5)(4)}{2}z^3 + \cdots,$$
$$\text{for } |z| < 1.$$

3.3 The function $f(z) = 1/z$ is analytic on the region $\mathbb{C} - \{0\}$. The largest open disc, centred at i, that will fit in this region is $D = \{z : |z - i| < 1\}$. So, by Taylor's Theorem, the Taylor series about i for f converges to $f(z)$ for $|z - i| < 1$.



The Taylor series is found by calculating the higher derivatives of $f(z) = 1/z$ at i. We have

$$\begin{aligned} f(z) &= z^{-1}, & &\text{so} & f(i) &= -i; \\ f'(z) &= -z^{-2}, & &\text{so} & f'(i) &= 1; \\ f^{(2)}(z) &= 2z^{-3}, & &\text{so} & f^{(2)}(i) &= 2i; \\ f^{(3)}(z) &= -(3)(2)z^{-4}, & &\text{so} & f^{(3)}(i) &= -3!; \\ f^{(4)}(z) &= (4)(3)(2)z^{-5}, & &\text{so} & f^{(4)}(i) &= -4!i. \end{aligned}$$

In general,

$$\frac{f^{(n)}(i)}{n!} = (-1)^n i^{-(n+1)} = -(i)^{n+1} = i^{n-1},$$
$$\text{for } n = 0, 1, 2, \ldots .$$

The Taylor series about i for f is therefore

$$\frac{1}{z} = -i + (z-i) + i(z-i)^2 - \cdots + i^{n-1}(z-i)^n + \cdots, \quad \text{for } |z-i| < 1,$$

$$= \sum_{n=0}^{\infty} i^{n-1}(z-i)^n, \quad \text{for } |z-i| < 1.$$

3.4 (a) For z in some open disc with centre 0

$$f(z) = f(-z) = \sum_{n=0}^{\infty} a_n(-z)^n = \sum_{n=0}^{\infty} (-1)^n a_n z^n.$$

By the uniqueness of Taylor series,

$$(-1)^n a_n = a_n, \quad \text{for } n = 0, 1, 2, \ldots,$$

so that $a_n = 0$, for n odd.

(b) For z in some open disc with centre 0

$$f(z) = -f(-z) = -\sum_{n=0}^{\infty} a_n(-z)^n = -\sum_{n=0}^{\infty} (-1)^n a_n z^n.$$

By the uniqueness of Taylor series,

$$(-1)^n a_n = -a_n, \quad \text{for } n = 0, 1, 2, \ldots,$$

so that $a_n = 0$, for n even.

Section 4

4.1 Here the Taylor series is the binomial series with $\alpha = -1/2$; that is,

$$1 + \binom{-1/2}{1} z + \binom{-1/2}{2} z^2 + \binom{-1/2}{3} z^3 + \cdots,$$

where

$$\binom{-1/2}{n} = \frac{(-1/2)(-3/2)(-5/2)\cdots(-1/2-(n-1))}{n!}$$

$$= (-1)^n \frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2n-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot 2n}.$$

The Taylor series is therefore

$$1 - \frac{1}{2}z + \frac{1\cdot 3}{2\cdot 4}z^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}z^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}z^4 - \cdots, \quad \text{for } |z| < 1.$$

We have left the coefficients in the form shown to indicate how the pattern continues.

4.2 (a) We know that

$$(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots, \quad \text{for } |z| < 1;$$

$$\operatorname{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \quad \text{for } |z| < 1.$$

So, by the Combination Rules,

$$h(z) = 3 + (3+1)z + (3-1/2)z^2 + (3+1/3)z^3 + \cdots$$

$$= 3 + 4z + 2\tfrac{1}{2}z^2 + 3\tfrac{1}{3}z^3 + \cdots, \quad \text{for } |z| < 1.$$

(b) We know that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C};$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

So, by the Combination Rules,

$$h(z) = 1 + z - \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} - \frac{z^6}{6!} - \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

4.3 (a) We know that

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots, \quad \text{for } z \in \mathbb{C};$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

Since we are interested only in terms up to z^3, it follows from the Product Rule that

$$h(z) = (1 + z + \frac{z^2}{2!} + \cdots)(z - \frac{z^3}{3!} + \cdots)$$

$$= z + z^2 + \left(-\frac{1}{3!} + \frac{1}{2!}\right)z^3 + \cdots$$

$$= z + z^2 + \frac{z^3}{3} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

(b) We know that

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C};$$

$$\operatorname{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \quad \text{for } |z| < 1.$$

Since we are interested only in terms up to z^3, it follows from the Product Rule, that

$$h(z) = \left(1 - \frac{z^2}{2!} + \cdots\right)\left(z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots\right)$$

$$= z - \frac{z^2}{2} + \left(\frac{1}{3} - \frac{1}{2!}\right)z^3 + \cdots$$

$$= z - \frac{z^2}{2} - \frac{z^3}{6} + \cdots, \quad \text{for } |z| < 1.$$

4.4 (a) We know from Problem 4.1 that

$$(1+w)^{-1/2} = 1 - \frac{1}{2}w + \frac{1\cdot 3}{2\cdot 4}w^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}w^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}w^4 - \cdots, \quad \text{for } |w| < 1.$$

Substituting $w = -z^2$, we obtain

$$h(z) = 1 + \frac{1}{2}z^2 + \frac{1\cdot 3}{2\cdot 4}z^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}z^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}z^8 + \cdots, \quad \text{for } |z| < 1.$$

(b) We know that

$$(1+w)^\alpha = 1 + \binom{\alpha}{1}w + \binom{\alpha}{2}w^2 + \binom{\alpha}{3}w^3 + \cdots, \quad \text{for } |w| < 1.$$

Substituting $w = z - 1$, we obtain

$$z^\alpha = 1 + \binom{\alpha}{1}(z-1) + \binom{\alpha}{2}(z-1)^2 + \binom{\alpha}{3}(z-1)^3 + \ldots,$$

$$= 1 + \alpha(z-1) + \frac{\alpha(\alpha-1)}{2}(z-1)^2 + \cdots, \quad \text{for } |z-1| < 1.$$

4.5 We know that

$$\exp w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \frac{w^4}{4!} + \cdots, \qquad \text{for } w \in \mathbb{C};$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \qquad \text{for } z \in \mathbb{C}.$$

Since we are interested only in terms up to z^5, we substitute $w = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$, to obtain the Taylor series about 0 for h, by the Composition Rule:

$$\begin{aligned} h(z) &= 1 + (z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots) \\ &\quad + \frac{(z - \frac{z^3}{3!} + \cdots)^2}{2!} + \frac{(z - \frac{z^3}{3!} + \cdots)^3}{3!} \\ &\quad + \frac{(z - \cdots)^4}{4!} + \frac{(z - \cdots)^5}{5!} + \cdots \\ &= 1 + z + \frac{1}{2!}z^2 + (-\frac{1}{3!} + \frac{1}{3!})z^3 \\ &\quad + (-\frac{2}{3!2!} + \frac{1}{4!})z^4 + (\frac{1}{5!} - \frac{3}{3!3!} + \frac{1}{5!})z^5 + \cdots \\ &= 1 + z + \frac{1}{2}z^2 - \frac{1}{8}z^4 - \frac{1}{15}z^5 + \cdots, \qquad \text{for } |z| < r, \end{aligned}$$

where r is some positive number. (Since h is entire, this representation must in fact hold for all $z \in \mathbb{C}$.)

4.6 The function h is a primitive of the function $f(z) = (1 - z^2)^{-1/2}$ on $\{z : |z| < 1\}$. Furthermore we know, from Problem 4.4(a), that, for $|z| < 1$,

$$\begin{aligned} (1 - z^2)^{-1/2} &= 1 + \frac{1}{2}z^2 + \frac{1 \cdot 3}{2 \cdot 4}z^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}z^6 \\ &\quad + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}z^8 + \cdots. \end{aligned}$$

So, by the Integration Rule, we have

$$\begin{aligned} h(z) &= z + \frac{1}{2}\frac{z^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{z^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{z^7}{7} \\ &\quad + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}\frac{z^9}{9} + \cdots, \qquad \text{for } |z| < 1. \end{aligned}$$

The constant of integration is zero since $h(0) = \sin^{-1} 0 = 0$.

4.7 (a) $\sum_{k=0}^{\infty} b_{k+m+1}(z - \alpha)^k$

(b) r

(c) analytic

(d) $\sum_{k=0}^{\infty} d_k(z - \alpha)^{k+m+1}$

(e) uniqueness

(f) zero

Section 5

5.1 (a) Since $f(z) = z^3(z - 1)^4(z + 2) = 0$ when $z = 0, 1$, or -2, it follows that f has zeros at 0, 1, and -2.

Now

$f(z) = z^3 \times$ (a function that is analytic but non-zero at 0);
$f(z) = (z - 1)^4 \times$ (a function that is analytic but non-zero at 1);
$f(z) = (z + 2) \times$ (a function that is analytic but non-zero at -2).

So f has a zero of order 3 at 0, a zero of order 4 at 1, and a simple zero at -2.

(b) Here

$f(z) = (z - 3) \times$ (a function that is analytic but non-zero at 3)

So f has a zero of order 1 at 3.

(c) Since $f(z) = (z^2 + 9)^3 e^{-z}/(z^2 + 4) = 0$ when $z = 3i$, or $-3i$, it follows that f has zeros at $3i$ and $-3i$.

Now

$f(z) = (z - 3i)^3 \times$ (a function that is analytic but non-zero at $3i$);
$f(z) = (z + 3i)^3 \times$ (a function that is analytic but non-zero at $-3i$).

So f has zeros of order 3 at $3i$ and at $-3i$.

5.2 (a) Although we can factor out z^4, that still leaves $\sin 2z$ which is zero at 0. So, using the Taylor series about 0 for sin, we obtain

$$\begin{aligned} f(z) &= z^4 \sin 2z \\ &= z^4 \left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \cdots\right) \\ &= z^5 \left(2 - \frac{2^3 z^2}{3!} + \frac{2^5 z^4}{5!} - \cdots\right) \\ &= z^5 \times \text{(a function that is analytic but non-zero at 0).} \end{aligned}$$

Thus f has a zero of order 5 at 0.

(b) Although we can factor out z^2, that still leaves $\cos z - 1$ which is zero at 0. So, using the Taylor series about 0 for cos, we obtain

$$\begin{aligned} f(z) &= z^2(\cos z - 1) \\ &= z^2 \left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) \\ &= z^4 \left(-\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \cdots\right) \\ &= z^4 \times \text{(a function that is analytic but non-zero at 0).} \end{aligned}$$

Thus f has a zero of order 4 at 0.

(c) In order to find the order of the zero at 0 we use the Taylor series about 0 for sin to obtain

$$\begin{aligned} f(z) &= 6\sin(z^2) + z^2(z^4 - 6) \\ &= 6(z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \cdots) + (z^6 - 6z^2) \\ &= 6\left(\frac{z^{10}}{5!} - \frac{z^{14}}{7!} + \cdots\right) \\ &= z^{10}\left(\frac{6}{5!} - \frac{6z^4}{7!} + \cdots\right) \\ &= z^{10} \times \text{(a function that is analytic but non-zero at 0).} \end{aligned}$$

Thus f has a zero of order 10 at 0.

5.3 Since f and g are represented by the same Taylor series on D, it follows that $f(z) = g(z)$ for all $z \in D$. Also, D has a limit point in $\mathcal{R}$; indeed every point of D is such a point. The conditions of the Uniqueness Theorem are therefore satisfied, and so f agrees with g on $\mathcal{R}$.

5.4 By the Uniqueness Theorem, $f = g$ provided that f and g agree on a set S with a limit point (in $\mathbb{C}$).

(a) This condition is sufficient since S has a limit point (for example, 2) in $\mathbb{C}$.

(b) Here $S = \mathbb{N}$ has no limit points, so the condition may not be sufficient. In fact, the analytic functions $f(z) = \sin \pi z$ and $g(z) = 0$ agree on $\mathbb{N}$ but are not equal, so the condition is not sufficient.

(c) In this case S has limit point 0, so the condition is sufficient.

5.5 Let $g(z) = z$. Then the entire functions f and g agree on the set

$$S = \{1/n : n = 1, 2, \ldots\}.$$

Since S has limit point $0 \in \mathbb{C}$, we deduce, by the Uniqueness Theorem, that f agrees with g throughout $\mathbb{C}$. Hence $f(z) = z$, for $z \in \mathbb{C}$.

SOLUTIONS TO THE EXERCISES

Section 1

1.1 (a) $s_0 = i$; $s_1 = i + i = 2i$; $s_2 = 3i$; $s_3 = 4i$. In general, $s_n = (n+1)i$, $n = 0, 1, 2, \ldots$.

(b) $s_0 = i$; $s_1 = i + \dfrac{i}{10} = 1.1i$; $s_2 = 1.1i + \dfrac{i}{100} = 1.11i$;
$s_3 = 1.11i + \dfrac{i}{1000} = 1.111i$.
In general, $s_n = 1.11\ldots1i$, $n = 0, 1, 2, \ldots$, with n 1s after the decimal point.

(c) $s_0 = 1$; $s_1 = 1 + i$; $s_2 = 1 + i - 1 = i$;
$s_3 = 1 + i - 1 - i = 0$.
In general,

$$s_n = \begin{cases} 1, & \text{if } n = 0, 4, 8, \ldots, \\ 1+i, & \text{if } n = 1, 5, 9, \ldots, \\ i, & \text{if } n = 2, 6, 10, \ldots, \\ 0, & \text{if } n = 3, 7, 11, \ldots . \end{cases}$$

1.2 (a) The series $\sum\limits_{n=0}^{\infty} \dfrac{1}{\sqrt{2}}(1-i)^n$ is divergent because it is a geometric series with $a = 1/\sqrt{2}$ and $z = 1 - i$ (Theorem 1.2(b), $|z| = |1 - i| = \sqrt{2} > 1$).

(b) The series $\sum\limits_{n=0}^{\infty} \left(\dfrac{1+i}{2}\right)^n$ is convergent because it is a geometric series with $a = 1$ and $z = (1+i)/2$ (Theorem 1.2(a), $|z| = |(1+i)/2| = \sqrt{2}/2 < 1$). Also

$$\sum_{n=0}^{\infty} \left(\frac{1+i}{2}\right)^n = \frac{1}{1-(1+i)/2} = \frac{2}{1-i} = 1 + i.$$

(c) The series $\sum\limits_{n=0}^{\infty} \left(\dfrac{1-i}{\sqrt{2}}\right)^n$ is divergent because it is a geometric series with $a = 1$ and $z = (1-i)/\sqrt{2}$ (Theorem 1.2(b), $|z| = |(1-i)/\sqrt{2})| = 1$).

(d) The series $\sum\limits_{n=2}^{\infty} \dbinom{n}{2} i^n$ is divergent by the Non-null Test, because the sequence of terms

$$\left\{\binom{n}{2} i^n\right\} = \left\{\frac{n(n-1)}{2} i^n\right\} \text{ is not null.}$$

1.3 First note that

$$\begin{aligned} 2^{-n/2} \cos\frac{n\pi}{4} &= \operatorname{Re}\left(2^{-n/2} \exp\left(\frac{n\pi}{4} i\right)\right) \\ &= \operatorname{Re}\left(\frac{1}{\sqrt{2}} e^{\pi i/4}\right)^n \\ &= \operatorname{Re}\left(\frac{1+i}{2}\right)^n \end{aligned}$$

But by Exercise 1.2(b),

$$\sum_{n=0}^{\infty} \left(\frac{1+i}{2}\right)^n = 1 + i,$$

and so

$$\sum_{n=0}^{\infty} 2^{-n/2} \cos\frac{n\pi}{4} = \operatorname{Re}(1+i) = 1, \text{ by Theorem 1.5.}$$

1.4 (a) The series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+1}$ is absolutely convergent, by the Comparison Test, as $|1/(n^2+1)| \le 1/n^2$, for $n = 1, 2, \ldots$, and $\sum\limits_{n=1}^{\infty} 1/n^2$ is convergent by Theorem 1.3.

(b) The series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+i}$ is absolutely convergent, by the Comparison Test. Indeed

$$|1/(n^2+i)| \le 1/(n^2-1) \le 2/n^2, \quad \text{for } n = 2, 3, \ldots,$$

(note that a finite number of terms does not affect absolute convergence) and $\sum\limits_{n=1}^{\infty} 2/n^2$ is convergent by Theorem 1.3 and the Multiple Rule.

(c) The series $\sum\limits_{n=1}^{\infty} \dfrac{2^n - i}{n^2}$ is divergent. Indeed,

$$\begin{aligned} \left|\frac{z_{n+1}}{z_n}\right| &= \left|\frac{2^{n+1}-i}{2^n - i}\right| \cdot \left|\frac{n^2}{(n+1)^2}\right| \\ &= \left|\frac{2 - i/2^n}{1 - i/2^n}\right| \cdot \left|\frac{1}{(1+1/n)^2}\right| \\ &\to 2 > 1 \text{ as } n \to \infty, \end{aligned}$$

and so the series is divergent by the Ratio Test.

(d) The series $\sum\limits_{n=1}^{\infty} \dfrac{i^n}{n\sqrt{n}}$ is absolutely convergent, by the Comparison Test, as $\left|\dfrac{i^n}{n\sqrt{n}}\right| = \dfrac{1}{n^{3/2}}$, for $n = 1, 2, \ldots$, and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ is convergent by Theorem 1.3.

(e) The series $\sum\limits_{n=0}^{\infty} e^{n(i-1)}$ is absolutely convergent, by the Ratio Test, as

$$\begin{aligned} \left|\frac{z_{n+1}}{z_n}\right| &= \left|e^{(n+1)(i-1)-n(i-1)}\right| \\ &= \left|e^{i-1}\right| \\ &= e^{-1} < 1. \end{aligned}$$

(In fact, this series is a geometric series with common ratio e^{i-1} and $|e^{i-1}| = e^{-1} < 1$.)

Section 2

2.1 (a) $\sum\limits_{n=0}^{\infty} (-z)^n$ is a geometric series with common ratio $-z$. By Theorem 1.2(a), it converges when $|-z| < 1$, and diverges when $|-z| > 1$. Since

$$|-z| < 1 \iff |z| < 1$$

and

$$|-z| > 1 \iff |z| > 1,$$

the radius of convergence is 1 and the disc of convergence is $\{z : |z| < 1\}$.

We present briefer solutions for parts (b)–(d).

(b) $\sum_{n=0}^{\infty}(3iz)^n$ is a geometric series with common ratio $3iz$. It converges when $|3iz| < 1$ and diverges when $|3iz| > 1$. Hence the radius of convergence is $\frac{1}{3}$ and the disc of convergence is $\{z : |z| < \frac{1}{3}\}$.

(c) $\sum_{n=0}^{\infty}(3i - z)^n$ is a geometric series with common ratio $3i - z$. It converges when $|3i - z| < 1$ and diverges when $|3i - z| > 1$. Hence the radius of convergence is 1 and the disc of convergence is $\{z : |z - 3i| < 1\}$.

(d) $\sum_{n=0}^{\infty}(2z - i)^n$ is a geometric series with common ratio $(2z - i)$. It converges when $|2z - i| < 1$ and diverges when $|2z - i| > 1$. Hence the radius of convergence is $\frac{1}{2}$ and the disc of convergence is $\left\{z : |z - i/2| < \frac{1}{2}\right\}$.

(e) Here we use the Ratio Test. We have (with $z \neq 0$)

$$\begin{aligned}\left|\frac{z_{n+1}}{z_n}\right| &= \left|\frac{(n+1)z^{n+1}}{nz^n}\right| \\ &= \frac{n+1}{n}|z| \\ &= \left(1 + \frac{1}{n}\right)|z| \\ &\to |z| \text{ as } n \to \infty.\end{aligned}$$

So, by the Ratio Test, the series $\sum_{n=0}^{\infty} nz^n$ converges when $|z| < 1$ and diverges when $|z| > 1$.

Hence the radius of convergence is 1 and the disc of convergence is $\{z : |z| < 1\}$.

(f) We have (with $z \neq -1$)

$$\begin{aligned}\left|\frac{z_{n+1}}{z_n}\right| &= \left|\frac{(n+1)!(z+1)^{n+1}}{n!(z+1)^n}\right| \\ &= (n+1)|z+1| \\ &\to \infty \text{ as } n \to \infty.\end{aligned}$$

So, by the Ratio Test, the series $\sum_{n=1}^{\infty} n!(z+1)^n$ is divergent for all $z \neq -1$. Hence the radius of convergence is 0 and the disc of convergence is $\varnothing$.

(g) We have (with $z \neq 0$)

$$\begin{aligned}\left|\frac{z_{n+1}}{z_n}\right| &= \left|\frac{(z/(n+1))^{n+1}}{(z/n)^n}\right| \\ &= \frac{|z|n^n}{(n+1)^{n+1}} \\ &= \frac{|z|}{n+1}\left(\frac{n}{n+1}\right)^n \\ &\to 0 \text{ as } n \to \infty,\end{aligned}$$

since $(n/(n+1))^n < 1$.

So, by the Ratio Test, the series $\sum_{n=1}^{\infty}\left(\frac{z}{n}\right)^n$ converges for all z. Hence the radius of convergence is ∞ and the disc of convergence is $\mathbb{C}$.

(h) We have (with $z \neq \pi$)

$$\begin{aligned}\left|\frac{z_{n+1}}{z_n}\right| &= \left|\frac{(z-\pi)^{n+1}/(n+1)!}{(z-\pi)^n/n!}\right| \\ &= \frac{1}{n+1}|z-\pi| \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

So, by the Ratio Test, the series $\sum_{n=1}^{\infty}\frac{(z-\pi)^n}{n!}$ converges for all z. Hence the radius of convergence is ∞ and the disc of convergence is $\mathbb{C}$.

2.2 **(a)** Since $\sum_{n=0}^{\infty}(2z)^n$ is a geometric series with common ratio $2z$, the sum function is

$$f(z) = \frac{1}{1-2z} \quad \left(|z| < \tfrac{1}{2}\right).$$

The disc of convergence is $\{z : |z| < \frac{1}{2}\}$.

In parts (b)–(d), we use the fact that the geometric series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \tag{1}$$

has disc of convergence $\{z : |z| < 1\}$ and sum function f, where

$$f(z) = \frac{1}{1-z}, \quad \text{for } |z| < 1.$$

(b) Since $\sum_{n=0}^{\infty}(n+1)z^n = \sum_{n=1}^{\infty}nz^{n-1}$, the power series

$$\sum_{n=0}^{\infty}(n+1)z^n = 1 + 2z + 3z^2 + \cdots \tag{2}$$

may be obtained from the power series (1) by differentiating term by term.

Hence, by the Differentiation Rule, the power series (2) has radius of convergence 1 and sum function g, where

$$g(z) = f'(z) = \frac{1}{(1-z)^2}, \quad \text{for } |z| < 1.$$

(c) Since $\sum_{n=0}^{\infty}(n+1)(n+2)z^n = \sum_{n=2}^{\infty}n(n-1)z^{n-2}$, the power series

$$\sum_{n=0}^{\infty}(n+1)(n+2)z^n = 2 + 6z + 12z^2 + \cdots \tag{3}$$

may be obtained from the power series (1) by differentiating twice term by term.

Hence, by the Differentiation Rule (applied twice), the power series (3) has radius of convergence 1 and sum function g, where

$$g(z) = f''(z) = \frac{2}{(1-z)^3}, \quad \text{for } |z| < 1.$$

(d) Since $\sum_{n=1}^{\infty}\frac{z^n}{n} = \sum_{n=0}^{\infty}\frac{z^{n+1}}{n+1}$, the power series

$$\sum_{n=1}^{\infty}\frac{z^n}{n} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \tag{4}$$

may be obtained from the power series (1) by integrating term by term.

Hence, by the Integration Rule, the function

$$F(z) = \text{ constant} + \sum_{n=1}^{\infty} \frac{z^n}{n}$$

is a primitive of f on $\{z : |z| < 1\}$.

But any primitive of f on $\{z : |z| < 1\}$ has the form

$$z \longmapsto \text{constant} - \text{Log}(1-z).$$

On putting $z = 0$ and comparing the two forms of the primitive of f, we see that the power series (4) has sum function g, where

$$g(z) = -\text{Log}(1-z), \quad \text{for } |z| < 1.$$

Section 3

3.1 **(a)** Since the function $f(z) = \sinh 2z$ is entire, the Taylor series about 0 for f converges to $f(z)$ for all $z \in \mathbb{C}$. Now

$$\begin{aligned}
f(z) &= \sinh 2z, & &\text{so} & f(0) &= 0;\\
f'(z) &= 2\cosh 2z, & &\text{so} & f'(0) &= 2;\\
f^{(2)}(z) &= 4\sinh 2z, & &\text{so} & f^{(2)}(0) &= 0;\\
f^{(3)}(z) &= 8\cosh 2z, & &\text{so} & f^{(3)}(0) &= 8;\\
f^{(4)}(z) &= 16\sinh 2z, & &\text{so} & f^{(4)}(0) &= 0.
\end{aligned}$$

In general,

$$f^{(2n)}(0) = 0, \quad \text{for } n = 0, 1, 2, \ldots,$$
$$f^{(2n-1)}(0) = 2^{2n-1}, \quad \text{for } n = 1, 2, \ldots.$$

So

$$\sinh 2z = 2z + \frac{8z^3}{3!} + \frac{32z^5}{5!} + \cdots + \frac{2^{2n-1}z^{2n-1}}{(2n-1)!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

(Here we have followed the usual convention of giving a general term that ignores terms like $0z^2$.)

(b) Since the function $f(z) = z\sin z$ is entire, the Taylor series about 0 for f converges to $f(z)$ for all $z \in \mathbb{C}$. Now

$$\begin{aligned}
f(z) &= z\sin z, & &\text{so} & f(0) &= 0;\\
f'(z) &= z\cos z + \sin z, & &\text{so} & f'(0) &= 0;\\
f^{(2)}(z) &= -z\sin z + 2\cos z, & &\text{so} & f^{(2)}(0) &= 2;\\
f^{(3)}(z) &= -z\cos z - 3\sin z, & &\text{so} & f^{(3)}(0) &= 0;\\
f^{(4)}(z) &= z\sin z - 4\cos z, & &\text{so} & f^{(4)}(0) &= -4;\\
f^{(5)}(z) &= z\cos z + 5\sin z, & &\text{so} & f^{(5)}(0) &= 0.
\end{aligned}$$

In general,

$$f^{(2n)}(z) = (-1)^n z\sin z + (-1)^{n+1}2n\cos z, \quad \text{for } n = 0, 1, 2, \ldots,$$
$$f^{(2n+1)}(z) = (-1)^n z\cos z + (-1)^n(2n+1)\sin z, \quad \text{for } n = 0, 1, 2, \ldots;$$

so that

$$f^{(2n)}(0) = (-1)^{n+1}2n \quad \text{and} \quad f^{(2n+1)}(0) = 0.$$

So

$$z\sin z = z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} - \cdots + \frac{(-1)^{n+1}z^{2n}}{(2n-1)!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

(c) Since the function $f(z) = e^{iz}$ is entire, the Taylor series about $\pi/4$ for f converges for all $z \in \mathbb{C}$. Now

$$\begin{aligned}
f(z) &= e^{iz}, & &\text{so} & f(\pi/4) &= e^{i\pi/4};\\
f'(z) &= ie^{iz}, & &\text{so} & f'(\pi/4) &= ie^{i\pi/4};\\
f^{(2)}(z) &= i^2e^{iz}, & &\text{so} & f^{(2)}(\pi/4) &= i^2e^{i\pi/4};\\
f^{(3)}(z) &= i^3e^{iz}, & &\text{so} & f^{(3)}(\pi/4) &= i^3e^{i\pi/4};\\
f^{(4)}(z) &= i^4e^{iz}, & &\text{so} & f^{(4)}(\pi/4) &= i^4e^{i\pi/4}.
\end{aligned}$$

In general,

$$\begin{aligned}
f^{(n)}(\pi/4) &= i^n e^{i\pi/4}\\
&= i^n(1+i)/\sqrt{2}, \quad \text{for } n = 0, 1, 2, \ldots.
\end{aligned}$$

So

$$e^{iz} = \frac{1+i}{\sqrt{2}}\left(1 + i(z-\pi/4) - \frac{(z-\pi/4)^2}{2!} + \cdots + \frac{i^n(z-\pi/4)^n}{n!} + \cdots\right), \quad \text{for } z \in \mathbb{C}.$$

Section 4

4.1 **(a)** Using the Binomial Series with $\alpha = \frac{1}{2}$, we obtain

$$\begin{aligned}
(1+z)^{1/2} &= 1 + \tfrac{1}{2}z + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}z^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}z^3 + \cdots\\
&= 1 + \tfrac{1}{2}z - \tfrac{1}{8}z^2 + \tfrac{1}{16}z^3 - \cdots, \quad \text{for } |z| < 1.
\end{aligned}$$

(b) Using the result of part (a) as given and with z replaced by $-z$, and the Combination Rules, we obtain

$$\begin{aligned}
&(1+z)^{1/2} - (1-z)^{1/2}\\
&\quad = 1 + \tfrac{1}{2}z - \tfrac{1}{8}z^2 + \tfrac{1}{16}z^3 - \cdots\\
&\qquad - \left(1 - \tfrac{1}{2}z - \tfrac{1}{8}z^2 - \tfrac{1}{16}z^3 - \cdots\right)\\
&\quad = z + \tfrac{1}{8}z^3 + \cdots, \quad \text{for } |z| < 1.
\end{aligned}$$

(c) For $|z| < 1$, $1-z$, $1+z$ and $(1+z)^{-1}$ lie in the open right half-plane so, by the Logarithmic Identities (Theorem 5.1, *Unit A2*),

$$\begin{aligned}
\text{Log}\left(\frac{1-z}{1+z}\right) &= \text{Log}(1-z) + \text{Log}\left(\frac{1}{1+z}\right)\\
&= \text{Log}(1-z) - \text{Log}(1+z)\\
&= \left(-z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots\right)\\
&\quad - \left(z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots\right)\\
&= -2z - \tfrac{2}{3}z^3 - \cdots, \quad \text{for } |z| < 1.
\end{aligned}$$

(d) On replacing z by z^2 in the Taylor series for cos, we obtain

$$\begin{aligned}
z^3\cos(z^2) &= z^3\left(1 - \frac{(z^2)^2}{2!} + \frac{(z^2)^4}{4!} - \cdots\right),\\
&= z^3 - \frac{z^7}{2!} + \frac{z^{11}}{4!} - \cdots, \quad \text{for } z \in \mathbb{C}.
\end{aligned}$$

(e) By the Product Rule,

$$\begin{aligned}
(\sin z)(\cos z) &= \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)\\
&= z + \left(-\frac{1}{3!} - \frac{1}{2!}\right)z^3\\
&\quad + \left(\frac{1}{5!} + \left(\frac{1}{2!}\right)\left(\frac{1}{3!}\right) + \frac{1}{4!}\right)z^5 + \cdots\\
&= z - \tfrac{2}{3}z^3 + \tfrac{2}{15}z^5 - \cdots, \quad \text{for } z \in \mathbb{C}.
\end{aligned}$$

Alternatively,

$$\begin{aligned}
(\sin z)(\cos z) &= \tfrac{1}{2}\sin 2z\\
&= \tfrac{1}{2}\left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \cdots\right)\\
&= z - \tfrac{2}{3}z^3 + \tfrac{2}{15}z^5 - \cdots, \quad \text{for } z \in \mathbb{C}.
\end{aligned}$$

(f) By the Composition Rule, there is an $r > 0$ such that

$$\begin{aligned}
\operatorname{Log}(\cosh z) &= \operatorname{Log}\left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right) \\
&= \left(\frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots\right) \\
&\quad - \tfrac{1}{2}\left(\frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)^2 \\
&\quad + \tfrac{1}{3}\left(\frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)^3 - \cdots \\
&= \frac{z^2}{2} + \left(\frac{1}{4!} - \tfrac{1}{2}\cdot\left(\frac{1}{2!}\right)^2\right) z^4 \\
&\quad + \left(\frac{1}{6!} - \left(\frac{1}{2!}\right)\left(\frac{1}{4!}\right) + \tfrac{1}{3}\left(\frac{1}{2!}\right)^3\right) z^6 + \cdots \\
&= \frac{z^2}{2} - \frac{z^4}{12} + \frac{z^6}{45} - \cdots, \quad \text{for } |z| < r.
\end{aligned}$$

An explicit value for r is difficult to obtain.

(g) Since $\tanh z = f'(z)$, for $|z| < r$, where $f(z) = \operatorname{Log}(\cosh z)$ and r is the same as in part (f), it follows, from the Differentiation Rule applied to the series in part (f), that the Taylor series about 0 for tanh is

$$\tanh z = z - \frac{z^3}{3} + \frac{2z^5}{15} - \cdots, \quad \text{for } |z| < r.$$

Since tanh is analytic on $\{z : |z| < \pi/2\}$, we can take $r = \pi/2$.

4.2 (a) Replacing z by $2z$ in the Taylor series for sinh about 0, we obtain

$$\sinh 2z = (2z) + \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} + \cdots + \frac{(2z)^{2n-1}}{(2n-1)!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

(b)

$$\begin{aligned}
z \sin z &= z\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + \frac{(-1)^{n+1} z^{2n-1}}{(2n-1)!} + \cdots\right) \\
&= z^2 - \frac{z^4}{3!} + \frac{z^6}{5!} - \cdots + \frac{(-1)^{n+1} z^{2n}}{(2n-1)!} + \cdots, \quad \text{for } z \in \mathbb{C}.
\end{aligned}$$

Section 5

5.1 (a) The function $f(z) = z^2(z^2+4)^3$ may be written as

$$f(z) = z^2(z+2i)^3(z-2i)^3.$$

So f has a zero of order 2 at 0, and zeros of order 3 at $-2i$ and at $2i$.

(b) The function $f(z) = z \sin z$ has zeros at $k\pi$, for $k \in \mathbb{Z}$. The Taylor series about 0 for f is

$$\begin{aligned}
f(z) &= z\left(z - \frac{z^3}{3!} + \cdots\right) \\
&= z^2\left(1 - \frac{z^2}{3!} + \cdots\right).
\end{aligned}$$

So f has a zero of order 2 at 0.

The Taylor series about $k\pi$ for f is

$$\begin{aligned}
f(z) &= z \sin z \\
&= z(-1)^k \sin(z - k\pi) \quad \text{(using the hint)} \\
&= z(-1)^k \left((z-k\pi) - \frac{(z-k\pi)^3}{3!} + \cdots\right) \\
&= (z-k\pi) z (-1)^k \left(1 - \frac{(z-k\pi)^2}{3!} + \cdots\right), \quad k \in \mathbb{Z}.
\end{aligned}$$

So f has a zero of order 1 at $k\pi$, for $k \in \mathbb{Z} - \{0\}$.